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이학박사 학위논문

Universal m-gonal forms

(보편 m각수의 합)

2020년 2월

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수리과학부

박 다 윤

Universal m-gonal forms

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Universal m-gonal forms

A dissertation
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Abstract

In this thesis, we study the universal m-gonal forms.

In the Chapter 3, we study the γ_m which is a criterion for universality of a m-gonal form, i.e., the representability of every positive integer up to γ_m characterize the universality of a m-gonal form. We prove the growth of γ_m is exactly linear on m.

In the Chapter 4, we find the minimal rank r_m for universal m-gonal form and provide a concrete m-gonal form of the minimal rank for all m sufficiently large.

And the last Chapter, we find the maximal rank R_m for proper universal m-gonal form for all m sufficiently large. The proper universal m-gonal form is a universal m-gonal form for which without its last component the universality is broken.

Key words: Universal m-gonal form
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Chapter 1

Introduction

The number of total dots of regular m-gon with x dots for each side is x-th m-gonal number and numbers of the sequence are called as an m-gonal numbers. The x-th m-gonal number have formula

$$P_m(x) = \frac{(m-2)}{2}x^2 - \frac{(m-4)}{2}x \quad \text{where } x \in \mathbb{N}. \tag{1.0.1}$$

The polygonal number was firstly introduced in 2nd century in B.C. and has been studied for a long time by many mathematicians. Fermat famously, claimed that every positive integer can be written as a sum of at most m m-gonal numbers. The celebrated Lagrange's four square theorem in 1770 and the Guass's Eureka theorem in 1796 resolved his claim for $m = 4$ and $m = 3$, respectively. And finally Cauchy completed the proof of Fermat's claim in 1813.

On the other hand, for the formula $P_m(x)$ of (1.0.1) by allowing variables negative integers and zero, too, we may generalize the m-gonal number, i.e., we may define the generalized x-th m-gonal number as

$$P_m(x) = \frac{(m-2)}{2}x^2 - \frac{(m-4)}{2}x \quad \text{where } x \in \mathbb{Z}. \tag{1.0.2}$$

And we call a weighted sum of generaized m-gonal number

$$F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$$

an m-gonal form where $a_i \in \mathbb{N}$. In this thesis, without loss of generality, we always assume that the coefficients of an m-gonal form are in ascending order, i.e., $a_1 \leq a_2 \leq \cdots \leq a_n$. If the diophantine equation $F_m(\mathbf{x}) = N$ has an integer solution $\mathbf{x} \in \mathbb{Z}^n$ for all $N \in \mathbb{N}$, then we say that the m-gonal form $F_m(\mathbf{x})$ is universal.

Liouville determined every universal ternary triangular forms in 1982. And 55 square forms(i.e., positive definite universal diagonal quadratic forms) was found by Ramanujan in the early 20th century. And Dickson confirmed the Ramanujan's list except one among them which is indeed not universal. More generally, Willerding additionally classified all positive definite non-diagonal universal quadratic forms.

On the other hand, in 1993, Conway and Schneeberger announced an amazing result called as the 15-theorem about the universality of quadratic form which states that the representability of only $1, 2, 3, 5, 6, 7, 10, 14$, and 15 guarantees the universality, i.e., representability of every positive integer by a quadratic form. Since a square form(4-gonal form) is just a diagonal quadratic form we may get that the universality of square form is also characterized by the representability of only finitely many positive integers up to 15 as a corollary of the 15-Theorem. And then one may naturally wonder whether these kinds of finiteness theorem holds for any m-gonal form. As an answer of the question, Kane and Liu [10] showed that there exists (unique and minimal) $\gamma_m \in \mathbb{N}$ for which if an m-gonal form represents every positive integer up to γ_m then the m-gonal form is universal, i.e., such a finiteness theorem always exists for any m. There are some explicitly calculated γ_m for some small m. Bosma and Kane [3] showed $\gamma_3 = 8$. The 15-Theorem deduces $\gamma_4 = 15$. Ju [8] showed $\gamma_5 = 109$. Since any hexagonal number is a triangular number and any triangular number is a hexagonal number, one can get $\gamma_3 = \gamma_6 = 8$. Ju and Oh [9] proved that $\gamma_8 = 60$.

The growth of γ_m was firstly questioned by Kane and Liu who proved that for $m \geq 3$ and every $\epsilon > 0$, there exists an absolute (effective) constant

C_ϵ such that $m-4 \le \gamma_m \ll C_\epsilon m^{7+\epsilon}$. In this thesis, we prove that there is an absolute constant C such that for any $m \ge 3$,

$$m-4 \le \gamma_m \le C(m-2)$$

holds by using the arithmetic theory of quadratic form, which implies that the growth of γ_m is exactly linear on m. Furthermore, with this result, we survey the rank of universal m-gonal forms for all m sufficiently large.

Chapter 2

Preliminaries

In this chapter, we introduce some definitions and well-known results which are used in throughout the thesis. Especially, we state the Jordan splitting and Local-to-Global Principle on quadratic lattices.

2.1 Quadratic Form

We call a composite object (V, Q) consisting a vector space V over an abstract field F of characteristic not 2, a symmetric bilinear form B on V, i.e., a mapping

$$B : V \times V \to F$$

satisfying

$$\begin{cases} B(x + \alpha y, z) = B(x, z) + \alpha B(y, z) \\ B(x, y) = B(y, x) \end{cases}$$

for all $x, y, z \in V$ and $\alpha \in F$ and the corresponding quadratic map Q defined by

$$Q(x) := B(x, x)$$

as a quadratic space. And we usually write simply V instead of (V, Q) if there is no confusing the adopted quadratic form Q on V. When F is a totally real number field, we call a quadratic space V over F a *positive definite quadratic space* if $Q(v)$ are totally positive for all non zero vectors $v \in V$. Throughout this thesis, we always assume that the field F is rational number field $\mathbb{Q}$ or its p-adic completion $\mathbb{Q}_p$.

For a basis $\{v_1, \cdots, v_n\}$ for V, we call a symmetric matrix

$$(B(v_i, v_j)) \in M_n(F)$$

as a *gram-matrix* of the quadratic space V. Then the gram-matrix would be unique up to similarity. So the determinant of gram-matrix of V is unique up to non-zero square. We say that V is regular if the determinant of its gram-matrix is not zero in $F/(F^\times)^2$. We usually identify a quadratic space with its gram-matrix and write

$$V \cong (B(v_i, v_j)).$$

Especially, for an orthogonal basis $\{v_1, \cdots, v_n\}$ for V, i.e., $B(v_i, v_j) = 0$ for all $1 \le i < j \le n$, we write

$$V \cong \langle Q(v_1) \rangle \perp \langle Q(v_2) \rangle \perp \cdots \perp \langle Q(v_n) \rangle .$$

For two quadratic spaces V and W over F, we say that V *is represented by* W *(or* W *represents* V*)* if there is an injective linear transformation

$$\sigma : V \to W$$

satisfying $B_V(x, y) = B_W(\sigma(x), \sigma(y))$ for all $x, y \in V$ where B_V and B_W are symmetric bilinear forms on V and W, respectively and in this case, we use the notation $V \longrightarrow W$. And we say that V *is isometric to* W if there is a bijective linear transformation

$$\sigma : V \to W$$

satisfying $B_V(x, y) = B_W(\sigma(x), \sigma(y))$ for all $x, y \in V$ and in this case, we use the notation $V \cong W$.

Let R be the ring of integers of F and let L be an R-module in a quadratic space (V, Q) over F with $B(L, L) \subseteq R$. We call the above (L, Q) or simply L as a *(quadratic) lattice over* R or simply R-lattice. Similarly, when

$\{v_1, \cdots, v_n\}$ is an R-basis for L, we write

$$L \cong (B(v_i, v_j))$$

and especially if $B(v_i, v_j) = 0$ for all $1 \le i < j \le n$, then write

$$L \cong \langle Q(v_1) \rangle \perp \langle Q(v_2) \rangle \perp \cdots \perp \langle Q(v_n) \rangle .$$

For two quadratic lattices L and K over R, we say that L *is represented by* K *(or L represents K)* if there is an injective linear transformation

$$\sigma : L \to K$$

satisfying $B_L(x, y) = B_K(\sigma(x), \sigma(y))$ for all $x, y \in L$ where B_L and B_K are symmetric bilinear forms on L and K, respectively and in this case, we use the notation $L \longrightarrow K$. And we say that L *is isometric to* K if there is a bijective linear transformation

$$\sigma : L \to K$$

satisfying $B_L(x, y) = B_K(\sigma(x), \sigma(y))$ for all $x, y \in L$ and in this case, we use the notation $L \cong K$.

For a sublattice L_1 of a quadratic R-lattice L, if there is a sublattice L_2 of L with

$$\begin{cases} L = L_1 \oplus L_2 \\ B(L_1, L_2) = 0, \end{cases}$$

then we say that the sublattice L_1 *splits* L and in this case, we write

$$L = L_1 \perp L_2.$$

Moreover, if $L = L_1 \oplus \cdots \oplus L_r$ with $B(L_i, L_j) = 0$ for all $1 \le i < j \le r$, we write

$$L = L_1 \perp \cdots \perp L_r$$

and say that L is the orthogonal sum of the sublattices $L_1, \cdots, L_r$.

We denote

$$\mathfrak{s}L$$

as the ideal generated by $B(L,L) = \{B(x,y)|x,y \in L\}$ and call *scale of* L. And define the *norm* of L

$$\mathfrak{n}L$$

as the ideal generated by $Q(L) = \{Q(x)|x \in L\}$. For an R-basis $\{v_1, \cdots, v_n\}$ for L, we define a *discriminant of* L

$$dL$$

as the determinant of its gram-matrix $(B(v_i, v_j))$. So the discriminant of L would be unique up to unit square. And when $dL \neq 0$, we call L is *regular*. We define the *volume* of L

$$\mathfrak{v}L$$

to be the ideal generated by dL.

And then one may observe that

$$\begin{cases} 2\mathfrak{s}L \subseteq \mathfrak{n}L \subseteq \mathfrak{s}L \\ \mathfrak{v}L \subseteq (\mathfrak{s}L)^n. \end{cases}$$

Especially, when L actually satisfies $\mathfrak{v}L = (\mathfrak{s}L)^n$, we call L as $(\mathfrak{s}L-)$*modular*. Note that the lattice Rv of rank 1 with $Q(v) \neq 0$ is always $(Q(v)R-)$modular.

Proposition 2.1.1. *Let L be a R-lattice in a quadratic space V and L_1 is an $\mathfrak{a}$-modular sublattice of L. Then L_1 splits L if and only if $B(L_1, L) \subseteq \mathfrak{a}$.*

Proof. See 82:15 in [11]. □

2.2 Jordan Splitting

Theorem 2.2.1. *For any lattice L over a ring of p-adic integer $\mathbb{Z}_p$, L has the following splitting*

$$L = L_1 \perp L_2 \perp \cdots \perp L_t$$

with $\begin{cases} L_i \text{ are modular for all } 1 \le i \le t \\ \mathfrak{s}L_1 \supsetneq \mathfrak{s}L_2 \supsetneq \cdots \supsetneq \mathfrak{s}L_t. \end{cases}$

We call this kind of splitting in Theorem 2.2.1 as a *Jordan splitting of* L. In the follwing Theorem 2.2.2, we see some relations between two Jordan splittings of L.

Theorem 2.2.2. *Let L be a quadratic lattice over $\mathbb{Z}_p$ and let*

$$L = L_1 \perp \cdots \perp L_t, \quad L = K_1 \perp \cdots \perp K_T$$

be two Jordan splittings of L. Then $t = T$. And for $1 \le \lambda \le t$ we have

(1) $\mathfrak{s}L_\lambda = \mathfrak{s}K_\lambda$, $\dim L_\lambda$=$\dim K_\lambda$.

(2) $\mathfrak{s}L_\lambda = \mathfrak{n}L_\lambda$ *if and only if* $\mathfrak{s}K_\lambda = \mathfrak{n}K_\lambda$

Proof. See 91:9 in [11]. □

For two quadratic $\mathbb{Z}_p$-lattices L and K, let

$$L = L_1 \perp \cdots \perp L_t, \quad K = K_1 \perp \cdots \perp K_T$$

be Jordan splittings of L and K, respectively. We say that these Jordan splittings are of the same type if $t = T$ and for all $1 \le \lambda \le t$, we have

$$\begin{cases} \mathfrak{s}L_\lambda = \mathfrak{s}K_\lambda, \quad \dim L_\lambda\text{=}\dim K_\lambda \\ \mathfrak{s}L_\lambda = \mathfrak{n}L_\lambda \text{ if and only if } \mathfrak{s}K_\lambda = \mathfrak{n}K_\lambda. \end{cases}$$

And then by Theorem 2.2.2, we may say that any two Jordan splittings of a quadratic lattice are of the same type.

Proposition 2.2.3. *Let L be a $p^r\mathbb{Z}_p-$modular quadratic lattice over $\mathbb{Z}_p$ where p is an odd prime. Then there is a unit $\epsilon \in \mathbb{Z}_p^\times$ such that*

$$L \cong \langle p^r \rangle \perp \langle p^r \rangle \perp \cdots \perp \langle p^r \rangle \perp \langle \epsilon p^r \rangle .$$

Proof. See 92:1 in [11]. □

Remark 2.2.4. *From the Theorem 2.2.1 and Proposition 2.2.3, we may see that a local lattice L over $\mathbb{Z}_p$ where p is an odd prime is diagonalizable. But generally a $\mathbb{Z}_2$-lattice may not always be diagonalizable.*

Corollary 2.2.5. *Let L be a $p^r\mathbb{Z}_p-$modular lattice over $\mathbb{Z}_p$ where p is an odd prime. Then*

(1) $Q(L) \supseteq p^r\mathbb{Z}_p^\times$ *if* $rankL = 2$.

(2) $Q(L) \supseteq p^r\mathbb{Z}_p$ *if* $rankL \geq 3$.

Theorem 2.2.6. *For a $\mathbb{Z}_p$-lattice L where p is an odd prime, let*

$$L = L_1 \perp \cdots, \perp L_t, \quad L = K_1 \perp \cdots, \perp K_t$$

be Jordan splittings of L. Then

$$dL_\lambda = dL_\lambda$$

for all $1 \leq \lambda \leq t$.

Proof. See 92:2 in [11]. □

Theorem 2.2.7. *Let L and K be isometric lattices over $\mathbb{Z}_p$ where p is an odd prime. Suppose there are splittings*

$$L = L_1 \perp L_2 \quad \text{and} \quad K = K_1 \perp K_2$$

with L_1 isometric to K_1. Then L_2 is isometric to K_2.

Proof. See 92:3 in [11]. □

Remark 2.2.8. *We call the property in Theorem 2.2.7 as a cancellation law. It makes calculating an orthogonal complement of a non-dyadic local lattice very easy.*

2.3 Local-to-Global Principle

Theorem 2.3.1. *Let U and V be regular quadratic spaces over $\mathbb{Q}$. Then U is represented by V if and only if $U \otimes \mathbb{Q}_p$ is represented by $V \otimes \mathbb{Q}_p$ for all primes p.*

Proof. See 66:3 in [11]. □

There is very effective criterion(cf. Theorem 63:21 in [11]) to determine the representability of quadratic spaces over local field. So with the Theorem 2.3.1 we may easily determine the representability of quadratic spaces over global field too. On the other hand, the criterion in Theorem 2.3.1 does not hold for quadratic lattice, i.e., for two regular quadratic lattices L and K over $\mathbb{Z}$,

$$L \otimes \mathbb{Z}_p \longrightarrow K \otimes \mathbb{Z}_p \text{ for every prime } p \text{ does not imply } L \longrightarrow K.$$

And generally determining whether a global quadratic lattice L over $\mathbb{Z}$ is represented by other global quadratic lattice K over $\mathbb{Z}$ is a hard problem. As a breakthrough in solving the problem, there is a well known result called as HKK Theroem.

Theorem 2.3.2. *(HKK Theorem) Let K be a positive definite $\mathbb{Z}$-lattice of rank $k \geq 2l + 3$. There is a constant $c = c(K)$ such that K represents any positive defnite $\mathbb{Z}$-lattice L, provided that*

$$\begin{cases} \mu(L) \geq c \\ K \otimes \mathbb{Z}_p \textit{ represents } L \otimes \mathbb{Z}_p \textit{ for every prime } p \end{cases}$$

where $\mu(L) := \min\{Q(x) | x \in L\}$.

Proof. See Theorem 3 in [7]. □

Especially, for the case $l = 1$ in Theorem 2.3.2, we may get the following useful Corollary.

Corollary 2.3.3. *Let L be a quadratic $\mathbb{Z}$-lattice of $rank(L) \geq 5$. Then there is a constant $c(L) > 0$ such that L represents any integer n, provided that*

$$\begin{cases} n \geq c(L) \\ L \otimes \mathbb{Z}_p \text{ represents } n \text{ at every prime } p. \end{cases}$$

The above Corollary 2.3.3 says that a $\mathbb{Z}$-lattice L of $\mathrm{rank}(L) \geq 5$ represents every sufficiently large integer which is represented by L locally.

Any unexplained terminology and notation can be found in [11].

2.4 Escalator Tree

Definition 2.4.1. For a non-universal m-gonal form

$$F_m(\mathbf{x}) = \sum_{i=1}^{k} a_i P_m(x_i),$$

we call the smallest positive integer which is not represented by $F_m(\mathbf{x})$ the *truant of* $F_m(\mathbf{x})$. And an extension

$$\sum_{i=1}^{k+1} a_i P_m(x_i) = F_m(\mathbf{x}) + a_{k+1} P_m(x_{k+1}) \tag{2.4.1}$$

of $F_m(\mathbf{x})$ which represents the truant of $F_m(\mathbf{x})$ is called as an *escalation of* $F_m(\mathbf{x})$.

An *escalator m-gonal form* or simply *escalator* is an m-gonal form which can be obtained thorough an escalating process starting from $\emptyset$ following $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1}$ in (2.4.1).

The *escalator tree* is a rooted tree which is constructed by connecting an escalator $\sum\limits_{i=1}^{k} a_i P_m(x_i)$ and its extension $\sum\limits_{i=1}^{k+1} a_i P_m(x_i)$ where $a_k \leq a_{k+1}$ with root $\emptyset$. Namely, in the case that a node $\sum\limits_{i=1}^{k} a_i P_m(x_i)$ is not universal, the node have children all of its extensions $\sum_{i=1}^{k+1} a_i P_m(x_i)$ with $a_k \le a_{k+1}$. If a node is universal, then the node would be a leaf on the escalator tree. All escalator m-gonal forms would appear in the escalator tree of m-gonal form and the universalities would only happen on leaves, more precisely, every leaf, i.e., universal escalator would be a proper universal m-gonal form in the sense that without its last component the universality is broken. We may observe that every universal m-gonal form contains one of universal escalators as a subform.

Lemma 2.4.2. *For any m, there is (unique and minimal) $\gamma_m \in \mathbb{N}$ for which if an m-gonal form represents every positive integer up to γ_m, then the m-gonal form represents every positive integer.*

Proof. See [10]. □

Since the truant of an escalator would get bigger than the truant of its parent, following the Lemma 2.4.2, the truant of any escalator could not exceed γ_m. Hence the escalator tree would have no infinite branch. Indeed, the largest truant of the escalators on the finite tree would be the γ_m. Before we go on, we remark the following well known facts.

Theorem 2.4.3. *(1) γ_m always exists and is finite for any $m \ge 3$.*

(2) $\gamma_m \ge m - 4$.

(3) In order for an m-gonal form $F_m(x_1, \cdots, x_n) = \sum_{i=1}^{n} a_i P_m(x_i)$ with $a_1 \le \cdots \le a_n$ to represent all positive integers up to $m-4$, the followings should be satisfied

$$\begin{cases} a_1 = 1 \\ a_{i+1} \le a_1 + \cdots + a_i + 1 \quad \text{if } a_1 + \cdots + a_i < m-4 \\ \sum_{i=1}^{n} a_i \ge m-4 \end{cases} \tag{2.4.2}$$

Proof. (1) is Lemma 2.4.2.
(2) is due to Guy [6].

(3) In order for $F_m(\mathbf{x})$ to represents 1, a_1 should be 1. If $a_{i+1} > a_1+\cdots+a_i+1$ when $a_1+\cdots+a_i < m-4$, then $F_m(\mathbf{x})$ may not represent $a_1+\cdots+a_i+1$ since the smallest m-gonal number is $P_m(-1)=m-3$ except $P_m(1)=1$ and $P_m(0)=0$. And finally we may get that if $\sum_{i=1}^n a_i < m-4$, then $F_m(\mathbf{x})$ does not represent all integers from $a_1+\cdots+a_n+1$ to $m-4$ by using the fact that the smallest m-gonal number is $P_m(-1)=m-3$ except $P_m(1)=1$ and $P_m(0)=0$ again. $\square$

Remark 2.4.4. *From $a_1=1$, inductively constructed a_k by following the condition*

$$a_{i+1} \le a_1+\cdots+a_i+1 \text{ for all } 1 \le i \le k-1 \tag{2.4.3}$$

can not exceed 2^{k-1}. So by the Theorem 2.4.3, when m is large, in order for an m-gonal form $F_m(\mathbf{x})$ to represent every positive integer up to only $m-4$, it is needed a lot of components more precisely, there is needed at least $\lceil \log_2(m-3) \rceil$ components. So the rank of a universal m-gonal form would be also greater than or equal to $\lceil \log_2(m-3) \rceil$.

We adopt the notations r_m(resp, R_m) for the minimal(resp, maximal) rank of universal escalators. And then from the above arguments, we may get that

$$\lceil \log_2(m-3) \rceil \le r_m \le R_m < \infty. \tag{2.4.4}$$

Furthermore, the r_m would be the minimal rank of universal m-gonal forms. And any universal m-gonal forms would contains a universal escalator as a subform. So we may induce that a universal m-gonal form contains a universal subform of the rank at most R_m.

Throughout this thesis, we consider γ_m, r_m, and R_m for all m sufficiently large. In Chapter 3, we show that the growth of γ_m is linear on m by proving that there is an absolute constant C for which

$$m-4 \le \gamma_m \le C(m-2) \tag{2.4.5}$$

for any $m \ge 3$. In Chapter 4 and Chapter 5, we find the exact r_m and R_m for all m sufficiently large, respectively. Finally, we additionally see that there is a universal escalator of rank n for each $r_m \le n \le R_m$.

Chapter 3

A finiteness theorem for universal m-gonal forms

In this chapter, we prove that there is an absolute constant C for which $m-4 \le \gamma_m \le C(m-2)$ for any $m \ge 3$.

3.1 Expression and Terminology

We denote by $\mathbf{e}_i$ the vector with 1 in the i-th coordinate and 0's elsewhere. We put $\alpha_n = \sum\limits_{i=1}^{n} \mathbf{e}_i \in \mathbb{Z}^n$ and especially when $n = 10$, we simply write $\alpha := \alpha_{10} \in \mathbb{Z}^{10}$.

For an n-ary m-gonal form $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$, we define an n-ary diagonal quadratic $\mathbb{Z}$-lattice $(\mathbb{Z}^n, Q_F)$ by

$$Q_F(x_1, \cdots, x_n) := a_1 x_1^2 + \cdots + a_n x_n^2$$

whose coefficients coincide with $F_m(\mathbf{x})$ and denote by B_F the corresponding symmetric bilinear mapping. And then we can express m-gonal form as the language of quadratic form theory as follows

$$F_m(\mathbf{x}) = \frac{m-2}{2}\{Q_F(\mathbf{x}) - B_F(\mathbf{x}, \alpha_n)\} + B_F(\mathbf{x}, \alpha_n) \tag{3.1.1}$$

for $\mathbf{x} \in \mathbb{Z}^n$. And now we define an $(n-1)$-ary $\mathbb{Z}$-sublattice (L_F, Q_F) of $(\mathbb{Z}^n, Q_F)$ by a hyperplane

$$L_F := \{(x_1, \cdots, x_n) \in \mathbb{Z}^n | B_F(\alpha_n, \mathbf{x}) = a_1x_1 + \cdots + a_nx_n = 0\}.$$

Then by (3.1.1), for $\mathbf{x} \in L_F$,

$$F_m(\mathbf{x}) = \frac{m-2}{2} Q_F(\mathbf{x}) \tag{3.1.2}$$

holds. For a vector $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}^n$, we define an $(n-2)$-ary $\mathbb{Z}$-sublattice $(L_{F,\beta}, Q_F)$ of (L_F, Q_F) by

$$L_{F,\beta} := \{(x_1, \cdots, x_n) \in L_F | B_F(\beta, \mathbf{x}) = a_1\beta_1x_1 + \cdots + a_n\beta_nx_n = 0\}.$$

And then we may observe that for $\mathbf{x} \in L_{F,\beta}$ and $r \in \mathbb{Z}$,

$$F_m(\mathbf{x}+r\beta) = \frac{m-2}{2}\{Q_F(\mathbf{x}) + r^2Q_F(\beta) - rB_F(\alpha_n, \beta)\} + rB_F(\alpha_n, \beta) \tag{3.1.3}$$

holds.

3.2 Motivation and Main Lemma

For a $\Gamma_m \in \mathbb{N}$, if any m-gonal form which represents every postive integer up to Γ_m is universal, then we may expect that the (unique, minimal) γ_m is bounded by the Γ_m. Indeed, among such Γ_m, the smallest one would be the real γ_m.

Throughout this chapter, we show that for sufficiently large $m > 2^{17}s$(s is a constant which will be determined after Lemma 3.2.1), any m-gonal form which represents every postive integer up to $C'(m-2)$ is universal, yielding one of upper bounds for γ_m is $C'(m-2)$(i.e., $\gamma_m \le C'(m-2)$) where $C'(\ge s)$ is an absolute constant which will be determined throughout this chapter. More precisely, we show that any m-gonal form which represents every positive integer up to $s(m-2)$ represents every positive integer greater than $C'(m-2)$ where $C' > s$. And then by defining

$$C := \max(\{\frac{\gamma_m}{m-2} | m \le 2^{17}s\} \cup \{C'\}),$$

we may assert that

$$m-4 \le \gamma_m \le C(m-2)$$

for any $m \ge 3$.

Lemma 3.2.1. *For $(a_1, \cdots, a_6) \in \mathbb{N}^6$, there is a constant $s(a_1, \cdots, a_6) \in \mathbb{N}$ which is dependent only on the 6-tuple $(a_1, \cdots, a_6) \in \mathbb{N}^6$ for which a senary m-gonal form*

$$F_m(\mathbf{x}) = \sum_{i=1}^{6} a_i P_m(x_i)$$

with coefficients $(a_1, \cdots, a_6)$ represents all the multiples of $s(a_1, \cdots, a_6)(m-2)$ for any $m \ge 3$.

Proof. Recall that for $\mathbf{x} \in L_F$,

$$F_m(\mathbf{x}) = \frac{m-2}{2} Q_F(\mathbf{x}). \tag{3.2.1}$$

On the other hand, since $\mathrm{rank}(L_F) = 5$, we have that the quadratic $\mathbb{Z}_p$-lattice $(L_F \otimes \mathbb{Z}_p, Q_F)$ is isotropic for all primes p. Thus for each prime p, we may take the minimal integer $e(p)$ for which $(L_F \otimes \mathbb{Z}_p, Q_F)$ represents every p-adic integer in $p^{e(p)}\mathbb{Z}_p$ and for almost all primes p the $e(p)$ would be 0. And then we may yield that (L_F, Q_F) represents all the multiples of $e(:= \prod p^{e(p)})$ locally. By Corollary 2.3.3, we see that there is a constant $2s(a_1, \cdots, a_6)$ which is a multiple of e for which the quadratic $\mathbb{Z}$-lattice (L_F, Q_F) represents all the multiples of $2s(a_1, \cdots, a_6)$. By applying the above results to the (3.2.1), we may conclude that $F_m(\mathbf{x})$ represents all the multiples of $s(a_1, \cdots, a_6)(m-2)$ when $\mathbf{x}$ runs through L_F. □

Remark 3.2.2. *For a sufficiently large $m \ge 2^{16}+3$, in order for an m-gonal form to represent every positive integer up to $m-4$, it is required more than $\lceil \log_2(m-3) \rceil$ (≤ 16) components and the tuple of first 16 coefficients should* *coincide with an element of the finite set*

$$\mathcal{A} := \{(a_1, \cdots, a_{16}) \in \mathbb{N}^{16} | a_1 = 1, a_i \le a_{i+1} \le a_1 + \cdots + a_i + 1 \text{ for all } 1 \le i \le 15\}$$

since the smallest m-gonal number is $m-3$ except $P_m(0) = 0$ and $P_m(1) = 1$.

Now let s be the least common multiple of all $s(a_{j_1}, \cdots, a_{j_6})$ where $(a_{j_1}, \cdots, a_{j_6})$ runs through the following finite set

$$\{(a_{j_1}, \cdots, a_{j_6}) \in \mathbb{N}^6 | (a_1, \cdots, a_{16}) \in \mathcal{A} \text{ with } 1 \le j_1 < \cdots < j_6 \le 16\}.$$

Then we may obtain the following Corollary.

Corollary 3.2.3. *For $m \ge 2^{16} + 3$, let $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$ be an m-gonal form which represents every positive integer up to $m - 4$. Then arbitrary senary subform*

$$f_m^*(x_{j_1}, \cdots, x_{j_6}) := a_{j_1} P_m(x_{j_1}) + \cdots + a_{j_6} P_m(x_{j_6})$$

where $1 \le j_1 < \cdots < j_6 \le 16$ of the sum of first 16 components $\sum_{i=1}^{16} a_i P_m(x_i)$ of $F_m(\mathbf{x})$ represents all the multiples of $s(m-2)$ where s is the constant given in Remark 3.2.2.

Proof. See Lemma 3.2.1 and Remark 3.2.2. □

Remark 3.2.4. *We will show that for each candidate for the first 16 coefficients $\mathbf{a} \in \mathcal{A}$ of m-gonal form which represent every positive integer up to $m-4$(or $s(m-2)$), there is a constant $C_{\mathbf{a}} > 0$ dependent only on $\mathbf{a}$ satisfying that*

$$\begin{aligned} &\text{for } m \ge 2^{17}s, \text{ having } \mathbf{a} \text{ as the first 16 coefficients } m\text{-gonal form} \\ &F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i) \text{ which represents every positive integer} \\ &\text{up to } m-4 (\text{or } s(m-2)) \text{ represents every integer greater than} \\ &C_{\mathbf{a}}(m-2) \text{ where } C_{\mathbf{a}} \ge s \text{ regardless of the rear part } \sum_{i=17}^{n} a_i P_m(x_i). \end{aligned} \tag{3.2.2}$$

If (3.2.2) holds, then we may say that for $m \geq 2^{17}s$, an m-gonal form $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$ having $\mathbf{a}$ as the first 16 coefficients is universal (i.e., represents every positive integer) if it is confirmed that $F_m(\mathbf{x})$ represents every positive integer up to only $C_{\mathbf{a}}(m-2)$. Let

$$C' := \max_{\mathbf{a} \in \mathcal{A}} C_{\mathbf{a}}.$$

Then for $m \geq 2^{17}s$, the universality of an m-gonal form may be well characterized by the representability of positive integers up to only $C'(m-2)$ since the first 16 coefficients of the m-gonal form under consideration is an element of the finite set $\mathcal{A}$. Furthermore, by defining the constant

$$C := \max(\{\frac{\gamma_m}{m-2} | 3 \leq m \leq 2^{17}s - 1\} \cup \{C_{\mathbf{a}} | \mathbf{a} \in \mathcal{A}\}),$$

we may generalize the result for any $m \geq 3$, nemely, we may get that

$$\gamma_m \leq C(m-2)$$

for any $m \geq 3$.

For each $\mathbf{a} \in \mathcal{A}$, the existence of such a constant $C_{\mathbf{a}}$ is shown throughout this chapter. Briefly, with the Corollary 3.2.3, by showing that

$$F_m(x_1, \cdots, x_n) - f_m^*(x_{j_1}, \cdots, x_{j_6}) = \sum_{i \in B} a_i P_m(x_i)$$

where $B = \{i | 1 \leq i \leq n, i \neq j_1, j_2, \cdots, j_6\}$ represents complete residues modulo $s(m-2)$ in the interval $[1, C_{\mathbf{a}}(m-2)]$ by suitably choosing $1 \leq j_1 < \cdots < j_6 \leq 16$. The following Lemma 3.2.5 is the main lemma in this chapter.

Lemma 3.2.5. *For $m \geq 2^{16} + 3$, let an m-gonal form*

$$F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$$

with $0 < a_1 \leq \cdots \leq a_n$ represents every positive integer up to $m-4$. If there *are a denary(10-ary) subform*

$$f_m(\mathbf{x}) := a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$$

where $1 \le i_1 < \cdots < i_{10} \le 16$ *of* $F_m(\mathbf{x})$ *and a vector*

$$\beta = (\beta_1, \cdots, \beta_{10}) \in \mathbb{Z}^{10}$$

with $B_f(\alpha, \beta) = a_{i_1}\beta_1 + \cdots + a_{i_{10}}\beta_{10} = 1$ *such that*

8-ary quadratic $\mathbb{Z}$*-lattice* $(L_{f,\beta}, Q_f)$ *is locally even universal,*

then we may take a constant

$$C_{a_1,\cdots,a_{16}} > 0$$

which is dependent only on the first 16 coefficients of $F_m(\mathbf{x})$ *from* a_1 *to* a_{16} *such that* $F_m(\mathbf{x})$ *represents all positive integers greater than or equal to* $C_{a_1,\cdots,a_{16}}(m-2)$ *regardless of the rear part* $\sum\limits_{i=17}^{n} a_iP_m(x_i)$.

Proof. In virtue of the Corollary 3.2.3, we may have that

$$f_m^*(x_{j_1}, \cdots, x_{j_6}) := a_{j_1}P_m(x_{j_1}) + \cdots + a_{j_6}P_m(x_{j_6})$$

represents every multiple of $s(m-2)$ where $\{j_1, \cdots, j_6\} = \{1, 2, \cdots, 16\} \setminus \{i_1, \cdots, i_{10}\}$. Thus in order to prove the lemma, it may be enough to show the existence of a constant $C_{\mathbf{a}} := C_{a_1,\cdots,a_{16}} > 0$ which is dependent only on the first 16 coefficients of $F_m(\mathbf{x})$ from a_1 to a_{16} for which

$$f_m(x_{i_1}, \cdots, x_{i_{10}}) + \sum_{i=17}^{n} a_iP_m(x_i)$$

represents complete residues modulo $s(m-2)$ in $[0, C_{\mathbf{a}}(m-2)]$. In other words, we define a constant $C_{\mathbf{a}}$ such that for each $N \in \mathbb{Z}/s(m-2)\mathbb{Z}$, there is an integer solution $(x_{i_1}, \cdots, x_{i_{10}}, x_{17}, \cdots, x_n) \in \mathbb{Z}^{n-6}$ for

$$\begin{cases} f_m(x_{i_1}, \cdots, x_{i_{10}}) + \sum_{i=17}^{n} a_i P_m(x_i) \equiv N \pmod{s(m-2)} \\ f_m(x_{i_1}, \cdots, x_{i_{10}}) + \sum_{i=17}^{n} a_i P_m(x_i) \le C_{\mathbf{a}}(m-2). \end{cases} \tag{3.2.3}$$

For each $0 \le r \le m-3$, by our assumption, we could have an integer solution

$$\mathbf{x}(r) = (x_1(r), \cdots, x_n(r)) \in \mathbb{Z}^n \tag{3.2.4}$$

for $F_m(\mathbf{x}(r)) = r$. Put

$$\begin{cases} r_1 := \sum_{i=1}^{16} a_i P_m(x_i(r)) \\ r_2 := \sum_{i=17}^{n} a_i P_m(x_i(r)). \end{cases}$$

Since the smallest non-zero m-gonal number is $P_m(1) = 1$ and the next one is $P_m(-1) = m-3$, all $x_i(r)$ of (3.2.4) would be 0 or 1 otherwise $r = m-3$. So we may obtain that

$$\begin{cases} 0 \le r_1 \le a_1 + \cdots + a_{16} \le 2^{16} - 1 \quad \text{or} \\ r_1 = m-3 \equiv -1 \pmod{m-2}. \end{cases}$$

Therefore r_1 may be congruent to one of an integer in

$$\{-1, 0, 1, 2, \cdots, 2^{16} - 1\}$$

modulo $m-2$. Let

$$r_1' := \begin{cases} -1 & \text{if } r_1 = m-3 \\ r_1 & \text{otherwise.} \end{cases}$$

Then $r_1 \equiv r_1' \pmod{m-2}$ and $-1 \le r_1' \le 2^{16} - 1$.

Recall that for all $\mathbf{x} \in L_{f,\beta}$ and $r_1 \in \mathbb{Z}$, we have

$$f_m(\mathbf{x} + r_1\beta) = \frac{m-2}{2}\{Q_f(\mathbf{x}) + r_1^2 Q_f(\beta) - r_1\} + r_1. \tag{3.2.5}$$

Since $L_{f,\beta}$ is locally even universal and $\text{rank}(L_{f,\beta}) = 8$ by the Corollary 2.3.3, we may yield that the quadraitc lattice $(L_{f,\beta}, Q_f)$ represents every sufficiently large even integer. So for each $-1 \le r_1' \le 2^{16}-1$ and $0 \le t \le s-1$, we could take a vector $\mathbf{x}_{t,r_1'} \in L_{f,\beta}$ such that

$$Q_f(\mathbf{x}_{t,r_1'}) \equiv 2t - r_1'^2 Q_f(\beta) + |r_1'| \pmod{2s} \tag{3.2.6}$$

because $2t - r_1'^2 Q_f(\beta) + |r_1'|$ is even integer. By applying the above result to (3.2.5), we may obtain that

$$f_m(\mathbf{x}_{t,r_1'} + r_1'\beta) \equiv t(m-2) + r_1 \pmod{s(m-2)}.$$

And then we may get that for $0 \le t \le s-1$ and $0 \le r \le m-3$,

$$f_m(\mathbf{x}_{t,r_1'} + r_1'\beta) + \sum_{i=17}^{n} a_i P_m(x_i(r)) \equiv t(m-2) + r \pmod{s(m-2)},$$

i.e., $f_m(x_{i_1}, \cdots, x_{i_{10}}) + \sum\limits_{i=17}^{n} a_i P_m(x_i)$ represents complete residues modulo $s(m-2)$. Let $C_{\mathbf{a}}$ denote the maximum of $s(2^{16}+1)$ integers

$$\frac{1}{2}\max\{Q_f(\mathbf{x}_{t,r_1'}) + r_1'^2 Q_f(\beta) - r_1' | 0 \le t \le s-1, -1 \le r_1' \le 2^{16}-1\} + 1$$

where $\mathbf{x}_{t,r_1'} \in L_{f,\beta}$ is a vector which we take for (3.2.6). Then for each $0 \le t \le s-1$ and $0 \le r \le m-3$, we may obtain that

$$\begin{aligned} f_m(\mathbf{x}_{t,r_1'} + r_1'\beta) + \sum_{i=17}^{n} a_i P_m(x_i(r)) &= \frac{m-2}{2}(Q_f(\mathbf{x}_{t,r_1'}) + r_1'^2 Q_f(\beta) - r_1') + r_1 + r_2 \\ &< \frac{m-2}{2}(Q_f(\mathbf{x}_{t,r_1'}) + r_1'^2 Q_f(\beta) - r_1') + (m-2) \\ &\le C_{\mathbf{a}}(m-2) \end{aligned}$$

holds. And by the construction, $C_{\mathbf{a}}$ is clearly dependent only on the first 16 coefficients of $F_m(\mathbf{x})$ from a_1 to a_{16}. This completes the proof. □

So now, our first duty is to check that for each $(a_1, \cdots, a_{16}) \in \mathcal{A}$, whether there are a 10-subtuple $(a_{i_1},\cdots,a_{i_{10}})$ and a vector $\beta\in\mathbb{Z}^{10}$ with $B_f(\alpha,\beta)=1$ where $f_m(\mathbf{x})=\sum_{k=1}^{10}a_{i_k}P_m(x_{i_k})$ for which the quadratic $\mathbb{Z}$-lattice $(L_{f,\beta},Q_f)$ is locally even universal, i.e., $(L_{f,\beta}\otimes\mathbb{Z}_p,Q_f)$ is even universal for every prime p.

On the other hand, we may observe that $(L_{f,\beta},Q_f)$ is the orthogonal complement $(\mathbb{Z}\alpha+\mathbb{Z}\beta)^{\perp}=\{\mathbf{x}\in\mathbb{Z}^{10}|B_f(\mathbf{x},\alpha)=B_f(\mathbf{x},\beta)=0\}$ in $(\mathbb{Z}^{10},Q_f)$. There are some well known criteria for determining the structure of an orthogonal complement of a quadratic lattice over local ring. Fortunately, the following proposition says that we may interchange the order of localization and orthogonal complementation.

Proposition 3.2.6. *Let L be a $\mathbb{Z}$-lattice and K be a sublattice of L. Then $(K\otimes\mathbb{Z}_p)^{\perp}=K^{\perp}\otimes\mathbb{Z}_p$ for all prime p.*

Proof. Clearly, $(K\otimes\mathbb{Z}_p)^{\perp}\supseteq K^{\perp}\otimes\mathbb{Z}_p$. Let $v\in(K\otimes\mathbb{Z}_p)^{\perp}=\{v\in L\otimes\mathbb{Z}_p|B(v,K\otimes\mathbb{Z}_p)=0\}$. Then v can be written as $v=\frac{1}{a}av$ where $a\in\mathbb{Z}$ with $p\nmid a$ and $av\in L$. Since $B(av,K)=0$, $v=\frac{1}{a}av\in K^{\perp}\otimes\mathbb{Z}_p$ □

3.3 Local structure $L_{f,\beta}\otimes\mathbb{Z}_p$

Throughout this section, we write

$$f_m(\mathbf{x})=a_{i_1}P_m(x_{i_1})+a_{i_2}P_m(x_{i_2})+\cdots+a_{i_{10}}P_m(x_{i_{10}})$$

and assume that

$$\beta\in\mathbb{Z}^{10}\text{ is a vector with }B_f(\alpha,\beta)=1.$$

In this section, we study some criteria to figure out the structure of local $\mathbb{Z}_p$-sublattice $L_{f,\beta}\otimes\mathbb{Z}_p$ of quadratic $\mathbb{Z}_p$-lattice $(\mathbb{Z}_p^{10},Q_f)$. The following proposition suggests an effective direction to discriminate whether $L_{f,\beta}$ is locally even universal.

Proposition 3.3.1. *Let p be an odd prime.*

(1) If there are 6 units of $\mathbb{Z}_p$(by admitting recursion) in $\{a_{i_1}, \cdots, a_{i_{10}}\}$, then $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal for any $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = 1$.

(2) If there are 5 units and 2 prime elements of $\mathbb{Z}_p$(by admitting recursion) in $\{a_{i_1}, \cdots, a_{i_{10}}\}$, then $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal for any $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = 1$.

Proof. Recall that $(L_{f,\beta}, Q_f)$ is the orthogonal complement $(\mathbb{Z}\alpha + \mathbb{Z}\beta)^\perp$ in $(\mathbb{Z}^{10}, Q_f)$.

(1) If $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ is unimodular, then we may use Proposition 2.1.1, Theorem 2.2.2 and Proposition 3.2.6 to obtain that

$$L_{f,\beta} \otimes \mathbb{Z}_p = (\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta)^\perp$$

contains a quaternary unimodular $\mathbb{Z}_p$-sublattice. This implies that $(L_{f,\beta} \otimes \mathbb{Z}_p, Q_f)$ is universal from Corollary 2.2.5.

Assume that $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ is not unimodular. From discriminant argument

$$d(\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta) = Q_f(\alpha)Q_f(\beta) - B_f(\alpha, \beta)^2 = Q_f(\alpha)Q_f(\beta) - 1 \in p\mathbb{Z}_p,$$

we may obtain that $Q_f(\alpha)$ is a unit over $\mathbb{Z}_p$. So we may diagonalize $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ as $\mathbb{Z}_p\alpha \perp \mathbb{Z}_p(\beta - Q_f(\alpha)^{-1}\alpha)$, namely,

$$\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta = \mathbb{Z}_p\alpha \perp \mathbb{Z}_p(\beta - Q_f(\alpha)^{-1}\alpha).$$

Since p is an odd prime, by Theorem 2.2.1, Theorem 2.2.2, and Proposition 2.2.3, there is an orthogonal $\mathbb{Z}_p$-basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{10}$ of $(\mathbb{Z}_p^{10}, Q_f)$ satisfying

$$\begin{cases} \mathbf{v}_1 = \alpha \\ Q_f(\mathbf{v}_2), \cdots, Q_f(\mathbf{v}_6) \in \mathbb{Z}_p^\times \end{cases}$$

because $\mathbb{Z}_p\alpha$ splits $(\mathbb{Z}_p^{10}, Q_f)$ by Proposition 2.1.1 and by our assumption that

$$(\mathbb{Z}^{10}, Q_f) \cong \langle a_{i_1} \rangle \perp \langle a_{i_2} \rangle \perp \cdots \perp \langle a_{i_{10}} \rangle$$

where at least 6 units in $\{a_{i_1}, \cdots, a_{i_{10}}\}$ over $\mathbb{Z}_p$. Since

$$B_f(\mathbf{v}_1, \beta - Q_f(\alpha)^{-1}\alpha) = 0,$$

we may write

$$\beta - Q_f(\alpha)^{-1}\alpha = p^{t_2}u_2\mathbf{v}_2 + p^{t_3}u_3\mathbf{v}_3 + \cdots + p^{t_{10}}u_{10}\mathbf{v}_{10}$$

where $t_i \in \mathbb{N}_0$ and $u_i \in \mathbb{Z}_p^\times$. Without loss of generality, we may assume that $0 \le t_2 \le t_3 \le \cdots \le t_6$. Define $M_\beta := \mathbb{Z}_p\mathbf{v}_2 + \cdots + \mathbb{Z}_p\mathbf{v}_6$, $\tilde{\beta} := \sum_{k=2}^{6} p^{t_k - t_2}u_k\mathbf{v}_k \in \mathbb{Z}_p^{10}$ and $\tilde{\gamma} := \mathbf{v}_2 \in \mathbb{Z}_p^{10}$.
If $Q_f(\tilde{\beta}) \in \mathbb{Z}_p^\times$, then we may see that $M_\beta \cap (\mathbb{Z}_p\tilde{\beta})^\perp$ is a quaternary unimodular $\mathbb{Z}_p$-lattice by using Proposition 2.1.1 and Theorem 2.2.2. So, by using the Proposition 3.2.6 and Corollary 2.2.5, we may see that $\mathbb{Z}_p$-sublattice $M_\beta \cap (\mathbb{Z}_p\tilde{\beta})^\perp$ of $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal, yielding the $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal.
If $Q_f(\tilde{\beta}) \in p\mathbb{Z}_p$, then we may see that $M_\beta \cap (\mathbb{Z}_p\tilde{\beta} + \mathbb{Z}_p\tilde{\gamma})^\perp$ is a ternary unimodular $\mathbb{Z}_p$-lattice by using the Proposition 2.1.1 and Theorem 2.2.2. So, by using the Proposition 3.2.6 and Corollary 2.2.5, we may see that the $\mathbb{Z}_p$-sublattice $M_\beta \cap (\mathbb{Z}_p\tilde{\beta} + \mathbb{Z}_p\tilde{\gamma})^\perp$ of $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal, yielding the $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal.

(2) If $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ is unimodular, then by again using Proposition 2.1.1, Theorem 2.2.2 and Proposition 3.2.6, we may see that $L_{f,\beta} \otimes \mathbb{Z}_p$ contains a ternary unimodular $\mathbb{Z}_p$-sublattice. This implies that $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal by Corollary 2.2.5.

Assume that $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ is not unimodular. From

$$d(\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta) = Q_f(\alpha)Q_f(\beta) - B_f(\alpha, \beta)^2 = Q_f(\alpha)Q_f(\beta) - 1 \in p\mathbb{Z}_p,$$

we may have that $Q_f(\alpha) \in \mathbb{Z}_p^\times$. So we may diagonalize $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ as $\mathbb{Z}_p\alpha \perp \mathbb{Z}_p(\beta - Q_f(\alpha)^{-1}\alpha)$, i.e.,

$$\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta = \mathbb{Z}_p\alpha \perp \mathbb{Z}_p(\beta - Q_f(\alpha)^{-1}\alpha).$$

Since p is an odd prime, by Theorem 2.2.1, Theorem 2.2.2, and Proposition 2.2.3, there may be an orthogonal $\mathbb{Z}_p$-basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{10}$ of $(\mathbb{Z}_p^{10}, Q_f)$ such that

$$\begin{cases} \mathbf{v}_1 = \alpha \\ Q_f(\mathbf{v}_2), Q_f(\mathbf{v}_3), Q_f(\mathbf{v}_4), Q_f(\mathbf{v}_5) \in \mathbb{Z}_p^\times \\ Q_f(\mathbf{v}_6), Q_f(\mathbf{v}_7) \in p\mathbb{Z}_p^\times \end{cases}$$

because $\mathbb{Z}_p\alpha$ splits $(\mathbb{Z}_p^{10}, Q_f)$ by Proposition 2.1.1 and by our assumption that

$$(\mathbb{Z}^{10}, Q_f) \cong \langle a_{i_1} \rangle \perp \langle a_{i_2} \rangle \perp \cdots \perp \langle a_{i_{10}} \rangle$$

with at least 5 units and 2 prime elements in $\{a_{i_1}, \cdots, a_{i_{10}}\}$ over $\mathbb{Z}_p$. Since $B_f(\mathbf{v}_1, \beta - Q_f(\alpha)^{-1}\alpha) = 0$, we may write

$$\beta - Q_f(\alpha)^{-1}\alpha = p^{t_2}u_2\mathbf{v}_2 + p^{t_3}u_3\mathbf{v}_3 + \cdots + p^{t_{10}}u_{10}\mathbf{v}_{10}$$

where $t_i \in \mathbb{N}_0$ and $u_i \in \mathbb{Z}_p^\times$. Without loss of generality, we may assume that $0 \le t_2 \le t_3 \le \cdots \le t_5$ and $t_6 \le t_7$. For the cases $t_2 \le t_6$, we may show that $L_{f,\beta} \otimes \mathbb{Z}_p$ contains a binary unimodular $\mathbb{Z}_p$-sublattice and a binary p-modular $\mathbb{Z}_p$-sublattice through a similar way used in the proof of (1), which yields that $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal.
If $t_2 > t_6$, then $L_{f,\beta} \otimes \mathbb{Z}_p$ contains a quaternary $\mathbb{Z}_p$-sublattice

$$\mathbb{Z}_p\mathbf{w}_2 + \mathbb{Z}_p\mathbf{w}_3 + \mathbb{Z}_p\mathbf{w}_4 + \mathbb{Z}_p\mathbf{w}_5$$

where $\mathbf{w}_k := \mathbf{v}_k - p^{t_k - t_6}u_k u_6^{-1} Q_f(\mathbf{v}_k) Q_f(\mathbf{v}_6)^{-1}\mathbf{v}_6$. Note that

$$\begin{cases} Q_f(\mathbf{w}_k) \equiv Q_f(\mathbf{v}_k) \not\equiv 0 \pmod p \\ B_f(\mathbf{w}_i, \mathbf{w}_j) \equiv 0 \pmod p \qquad\qquad \text{for } i \ne j. \end{cases} \tag{3.3.1}$$

By the modulus condition (3.3.1), we may induce that

$$d(\mathbb{Z}_p\mathbf{w}_2 + \mathbb{Z}_p\mathbf{w}_3 + \mathbb{Z}_p\mathbf{w}_4 + \mathbb{Z}_p\mathbf{w}_5) \in \mathbb{Z}_p^\times,$$

yielding the quaternary $\mathbb{Z}_p$-sublattice $\mathbb{Z}_p\mathbf{w}_2 + \mathbb{Z}_p\mathbf{w}_3 + \mathbb{Z}_p\mathbf{w}_4 + \mathbb{Z}_p\mathbf{w}_5$ of $L_{f,\beta} \otimes \mathbb{Z}_p$ is unimodular. So, $L_{f,\beta} \otimes \mathbb{Z}_p$ would be also (even) universal by Corollary 2.2.5. $\square$

Remark 3.3.2. *From now on, throughout this chapter, for each* $(a_1, \cdots, a_{16}) \in$

$\mathcal{A}$, we always take $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$ with

$$i_1 = 1,\ i_2 = 2, \cdots, i_7 = 7. \tag{3.3.2}$$

Then for all primes $p \geq 17$, since

$$\begin{cases} a_i < 2^i < 17 \leq p & \text{for } i \leq 5 \\ a_i < 2^i < 17^2 \leq p^2 & \text{for } i \leq 7, \end{cases}$$

$L_{f,\beta} \otimes \mathbb{Z}_p$ would be (even) universal for any $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = 1$ by Proposition 3.3.1. So under the condition (3.3.2), in order to take suitable remaining $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta}$ is locally even universal, we only need to pay attention to the primes p less than or equal to 13, i.e., it may be enough to check whether $L_{f,\beta} \otimes \mathbb{Z}_p$ are even universal for all primes $p \leq 13$ or not.

For each $\mathbf{a} \in \mathcal{A}$, we define $P(\mathbf{a})$ by the set of all odd primes satisfying none of the following two conditions by admitting recursion

$$\text{there are at least 6 units in } \{a_1, \cdots, a_7\} \text{ over } \mathbb{Z}_p \tag{3.3.3}$$

and

$$\text{there are 5 units and 2 primes in } \{a_1, \cdots, a_7\} \text{ over } \mathbb{Z}_p. \tag{3.3.4}$$

Note that for $\mathbf{a} \in \mathcal{A}$, we may have that $P(\mathbf{a}) \subseteq \{3, 5, 7, 11, 13\}$ by the above arguments. We define

$$\mathcal{A}(p) := \{\mathbf{a} \in \mathcal{A} | p \in P(\mathbf{a})\}$$

for each $p = 3, 5, 7, 11, 13$.

In order to determine every $\mathbf{a} \in \mathcal{A}$ satisfying that $(L_{f,\beta}, Q_f)$ is locally even universal for some $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta = (\beta_1, \cdots, \beta_{10}) \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = a_{i_1} \cdot \beta_1 + \cdots + a_{i_{10}} \cdot \beta_{10} = 1$, it may suffice to check the followings

(1) For each $\mathbf{a} \in \mathcal{A}(p) \setminus \bigcup_{p'>p} \mathcal{A}(p')$*, there are* $8 \le i_8 < i_9 < i_{10} \le 16$ *and* $\beta \in \mathbb{Z}^{10}$ *such that* $(L_{f,\beta} \otimes \mathbb{Z}_q, Q_f)$ *are even universal for all primes* $q \le p$ *for each* $p \in \{3, 5, 7, 11, 13\}$*.*

(2) For each $\mathbf{a} \in \mathcal{A} \setminus \bigcup_{p=3}^{13} \mathcal{A}(p) =: A(2)$*, there are* $8 \le i_8 < i_9 < i_{10} \le 16$ *and* $\beta \in \mathbb{Z}^{10}$ *such that* $(L_{f,\beta} \otimes \mathbb{Z}_2, Q_f)$ *is even universal.*

But we investigate each $\mathbf{a} \in \mathcal{A}(p) (\supseteq \mathcal{A}(p) \setminus \bigcup_{p'>p} \mathcal{A}(p'))$ *in* (1) *and each* $\mathbf{a} \in \mathcal{A}(\supseteq \mathcal{A} \setminus \bigcup_{p=3}^{13} \mathcal{A}(p) =: A(2))$ *in* (2)*.*

We now introduce a proposition which is mainly used to calculate the structure of an orthogonal complement $L_{f,\beta} \otimes \mathbb{Z}_p = (\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta)^{\perp}$ over non-dyadic local ring.

Proposition 3.3.3. *For an odd prime* p*, if* $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ *is unimodular and there are at least three units of* $\mathbb{Z}_p$ *in* $\{a_{i_1}, \cdots, a_{i_{10}}\}$*. By rearranging the order, suppose that* $a_{i_1}, a_{i_2}, a_{i_3} \in \mathbb{Z}_p^{\times}$*. Then* $L_{f,\beta} \otimes \mathbb{Z}_p$ *is isometric to*

$$\left\langle \frac{a_{i_1}a_{i_2}a_{i_3}}{Q_f(\alpha)Q_f(\beta) - 1} \right\rangle \perp \langle a_{i_4} \rangle \perp \cdots \perp \langle a_{i_{10}} \rangle .$$

Proof. It follows from Proposition 2.1.1, Theorem 2.2.2, Corollary 2.2.5, and Theorem 2.2.7. $\square$

Note that for any $\mathbf{a} \in \mathcal{A}$, $a_{i_1}(= a_1), a_{i_2}(= a_2)$, and $a_{i_3}(= a_3)$ are units over $\mathbb{Z}_p$ for all odd primes p except $p = 3$. In virtue of Proposition 3.3.3, we may easily figure out the exact structure of $L_{f,\beta} \otimes \mathbb{Z}_p$ when p is an odd prime and $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ is unimodular(we try to take a suitable vector β for which $\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta$ is unimodular for primes p to review based on the Remark 3.3.2). On the other hand, when $p = 2$, since the criterion in Proposition 3.3.3 does not hold, we need to find an even universal sublattice of $L_{f,\beta} \otimes \mathbb{Z}_2$ directly to claim that $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal. The following lemmas introduce some criteria which are used to claim that some $L_{f,\beta} \otimes \mathbb{Z}_2$ are even universal by showing that $L_{f,\beta} \otimes \mathbb{Z}_2$ contains an even universal $\mathbb{Z}_2$-sublattice.

Proposition 3.3.4. *For $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$ and $\beta = (\beta_1, \cdots, \beta_{10}) \in \mathbb{Z}^{10}$, suppose that there are distinct 3 indices $j_1, j_2, j_3 \in \{1, 2, \cdots, 10\}$ such that*

$$\begin{cases} \beta_{j_1} = \beta_{j_2} = \beta_{j_3} \\ a_{i_{j_1}} \equiv a_{i_{j_2}} \equiv a_{i_{j_3}} \equiv 1 \pmod 2. \end{cases}$$

(1) If $a_{i_{j_1}} \not\equiv a_{i_{j_2}} \pmod 4$ or $a_{i_{j_1}} \not\equiv a_{i_{j_3}} \pmod 4$, then $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal.

(2) If $a_{i_{j_1}} \equiv a_{i_{j_2}} \equiv a_{i_{j_3}} \pmod 4$, then $L_{f,\beta} \otimes \mathbb{Z}_2$ represents every 2-adic integer of odd order.

(3) If $a_{i_{j_1}} \equiv a_{i_{j_2}} \equiv a_{i_{j_3}} \pmod 4$ and there is an extra $j_4 \in \{1, 2, \cdots, 10\}$ with $\beta_{j_1} = \beta_{j_4}$ and $a_{i_{j_4}} \equiv 2 \pmod 4$, then $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal.

Proof. (1) $L_{f,\beta} \otimes \mathbb{Z}_2$ contains a $\mathbb{Z}_2$-sublattice

$$\mathbb{Z}_2(a_{i_{j_2}}\mathbf{e}_{j_1} - a_{i_{j_1}}\mathbf{e}_{j_2}) + \mathbb{Z}_2(a_{i_{j_3}}\mathbf{e}_{j_2} - a_{i_{j_2}}\mathbf{e}_{j_3})$$

which is isometric to even universal hyperplane

$$\mathbb{H} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This yields the claim.
(2) Since $L_{f,\beta} \otimes \mathbb{Z}_2$ contains a $\mathbb{Z}_2$-sublattice

$$\mathbb{Z}_2(a_{i_{j_2}}\mathbf{e}_{j_1} - a_{i_{j_1}}\mathbf{e}_{j_2}) + \mathbb{Z}_2(a_{i_{j_3}}\mathbf{e}_{j_2} - a_{i_{j_2}}\mathbf{e}_{j_3})$$

which is equivalent to

$$\mathbb{A} \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $\mathbb{A}$ represents every prime element in $\mathbb{Z}_2$. This yields the claim.
(3) $L_{f,\beta} \otimes \mathbb{Z}_2$ contains a $\mathbb{Z}_2$-sublattice

$$\mathbb{Z}_2(a_{i_{j_2}}\mathbf{e}_{j_1} - a_{i_{j_1}}\mathbf{e}_{j_2}) + \mathbb{Z}_2((a_{i_{j_3}} + a_{i_{j_4}})\mathbf{e}_{j_2} - a_{i_{j_2}}(\mathbf{e}_{j_3} + \mathbf{e}_{j_4}))$$

which is isometric to even universal hyperplane

$$\mathbb{H} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This yields the claim.

$\square$

Proposition 3.3.5. *For $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$ and $\beta = (\beta_1, \cdots, \beta_{10}) \in \mathbb{Z}^{10}$, suppose that $\beta_{j_1} = \beta_{j_2} = \cdots = \beta_{j_l}$ for all distinct $j_1, \cdots, j_l \in \{1, 2, \cdots, 10\}$, then the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ contains a $\mathbb{Z}$-sublattice which is isometric to*

$$\langle b_1 \rangle \perp \cdots \perp \langle b_{l-1} \rangle$$

where $b_h = a_{i_{j_{h+1}}}(a_{i_{j_1}} + a_{i_{j_2}} + \cdots + a_{i_{j_h}})(a_{i_{j_1}} + a_{i_{j_2}} + \cdots + a_{i_{j_{h+1}}})$ for $1 \leq h \leq l-1$.

Proof. Define

$$\mathbf{v}_h := a_{i_{j_{h+1}}}(\mathbf{e}_{j_1} + \mathbf{e}_{j_2} + \cdots + \mathbf{e}_{j_h}) - (a_{i_{j_1}} + a_{i_{j_2}} + \cdots + a_{i_{j_h}})\mathbf{e}_{j_{h+1}}$$

for $h = 1, 2, \cdots, l-1$. Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{l-1} \in L_{f,\beta} = (\mathbb{Z}\alpha + \mathbb{Z}\beta)^{\perp}$ and they are mutually orthogonal. Namely, the quadratic $\mathbb{Z}$-lattice $(L_{f,\beta}, Q_f)$ contains a $\mathbb{Z}$-sublattice

$$\mathbb{Z}\mathbf{v}_1 \perp \mathbb{Z}\mathbf{v}_2 \perp \cdots \perp \mathbb{Z}\mathbf{v}_{l-1}$$

with $Q_f(\mathbf{v}_h) = b_h$. This completes the proof. $\square$

Proposition 3.3.6. *For $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$ with $(a_{i_1}, a_{i_2}) = (1, 2)$ put $\beta := -\mathbf{e}_1 + \mathbf{e}_2 \in \mathbb{Z}^{10}$. Then $B_f(\alpha, \beta) = 1$ holds.*

(1) If there are at least three odd coefficients $a_{i_{j_1}}, a_{i_{j_2}}, a_{i_{j_3}} \in \{a_{i_3}, \cdots, a_{i_{10}}\}$, then $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal.

(2) For all distinct l indices $j_1, \cdots, j_l \in \{3, 4, \cdots, 10\}$, the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ contains a $\mathbb{Z}$-sublattice which is isometric to

$$\langle b_0 \rangle \perp \langle b_1 \rangle \perp \cdots \perp \langle b_{l-1} \rangle$$

where $a := a_{i_{j_1}} + a_{i_{j_2}} + \cdots + a_{i_{j_l}}$ *and*

$$b_h = \begin{cases} \left(\frac{2a}{\gcd(4,a)}\right)^2 + 2\left(\frac{a}{\gcd(4,a)}\right)^2 + a\left(\frac{-4}{\gcd(4,a)}\right)^2 & \text{if } h = 0 \\ a_{i_{j_{h+1}}}(a_{i_{j_1}} + \cdots + a_{i_{j_h}})(a_{i_{j_1}} + \cdots + a_{i_{j_{h+1}}}) & \text{if } h \geq 1. \end{cases}$$

Proof. (1) If $a_{i_{j_1}} \not\equiv a_{i_{j_2}} \pmod 4$ or $a_{i_{j_1}} \not\equiv a_{i_{j_3}} \pmod 4$, then by Proposition 3.3.4 (1), it is done.
If $a_{i_{j_1}} \equiv a_{i_{j_2}} \equiv a_{i_{j_3}} \pmod 4$, then the $\mathbb{Z}_2$-sublattice

$$\mathbb{Z}_2(2(a_{i_{j_1}} + a_{i_{j_2}} + a_{i_{j_3}})\mathbf{e}_1 + (a_{i_{j_1}} + a_{i_{j_2}} + a_{i_{j_3}})\mathbf{e}_2 - 4(\mathbf{e}_{j_1} + \mathbf{e}_{j_2} + \mathbf{e}_{j_3})) \\ \perp (\mathbb{Z}_2(a_{i_{j_2}}\mathbf{e}_{j_1} - a_{i_{j_1}}\mathbf{e}_{j_2}) + \mathbb{Z}_2(a_{i_{j_3}}\mathbf{e}_{j_2} - a_{i_{j_2}}\mathbf{e}_{j_3}))$$

of $L_{f,\beta} \otimes \mathbb{Z}_2$ is isometric to even universal $\mathbb{Z}_2$-lattice $\langle 6 \rangle \perp \mathbb{A}$. This completes the proof.

(2)The quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ contains a $\mathbb{Z}$-sublattice

$$\mathbb{Z}\mathbf{v}_1 \perp \mathbb{Z}\mathbf{v}_2 \perp \cdots \perp \mathbb{Z}\mathbf{v}_{l-1}$$

where $a := a_{i_{j_1}} + a_{i_{j_2}} + \cdots + a_{i_{j_l}}$ and

$$\mathbf{v}_h = \begin{cases} \frac{2a}{\gcd(4,a)}\mathbf{e}_1 + \frac{a}{\gcd(4,a)}\mathbf{e}_2 - \frac{4}{\gcd(4,a)}(\mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_l}) & \text{if } h = 0 \\ a_{i_{j_{h+1}}}(\mathbf{e}_{j_1} + \mathbf{e}_{j_2} + \cdots + \mathbf{e}_{j_h}) - (a_{i_{j_1}} + a_{i_{j_2}} + \cdots + a_{i_{j_h}})\mathbf{e}_{j_{h+1}} & \text{if } h \geq 1. \end{cases}$$

And $Q(\mathbf{v}_h) = b_h$. This completes the proof. □

Proposition 3.3.7. *Any* $\mathbb{Z}_2$*-lattice which is isometric to* $\langle b_1 \rangle \perp \langle b_2 \rangle \perp \langle b_3 \rangle \perp \langle b_4 \rangle$ *with* $(\nu_2(b_1), \nu_2(b_2), \nu_2(b_3), \nu_2(b_4)) \in \Delta_2$ *is even universal where* $\nu_2(n) := \max\{v \in \mathbb{N} : 2^v | n\}$ *and*

$$\Delta_2 := \{(1,1,1,1), (1,1,1,2), (1,1,2,2), (1,1,2,3), \\ (1,1,2,4), (1,2,2,3), (1,2,3,3), (1,2,3,4)\}.$$

Proof. The proof is omitted here. □

Remark 3.3.8. *For* $\mathbf{a} \in \mathcal{A}$*, if there are three odd coefficients in* $\{a_1, \cdots, a_{16}\}$*,* *we try to take suitable $a_{i_8}, a_{i_9}, a_{i_{10}} \in \{a_8, \cdots, a_{16}\}$ and a vector $\beta \in \mathbb{Z}^{10}$ satisfying the assumption of Proposition 3.3.4 to get a locally even universal quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$(of course by simultaneously considering other primes which are needed to review). On the other hand, in the case that for $\mathbf{a} \in \mathcal{A}$, the number of odd coefficients in $\{a_1, \cdots, a_{16}\}$ is less than three or there is difficulty in getting a suitable vector $\beta \in \mathbb{Z}^{10}$ for locally even universal $L_{f,\beta}$ satisfying the condition of Proposition 3.3.4 by other issues caused at other prime spots, the Proposition 3.3.5 may be an alternative guideline to take $a_{i_8}, a_{i_9}, a_{i_{10}} \in \{a_8, \cdots, a_{16}\}$ and a vector $\beta \in \mathbb{Z}^{10}$ for which $(L_{f,\beta}, Q_f)$ is locally even universal quadratic $\mathbb{Z}$-lattice. In such the case, we try to take $a_{i_8}, a_{i_9}, a_{i_{10}} \in \{a_8, \cdots, a_{16}\}$ and a vector $\beta \in \mathbb{Z}^{10}$ such that for some distinct $j_1, j_2, \cdots, j_l \in \{1, 2, \cdots, 10\}$*

$$\begin{cases} \beta_{j_1} = \beta_{j_2} = \cdots = \beta_{j_l} \\ \nu_2(a_{i_{j_1}}) = 0 \ \textit{and} \ (\nu_2(a_{i_{j_2}}), \cdots, \nu_2(a_{i_{j_5}})) \in \Delta_2 \end{cases} \tag{3.3.5}$$

holds. When (3.3.5) holds, by using the Proposition 3.3.5 and Proposition 3.3.7, we may induce that $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal.

Case I. $\mathcal{A}(\mathbf{13})$

In this case, we show that for each $\mathbf{a} \in \mathcal{A}(13)$, there are suitable $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta}$ is locally even universal. We have

$$\begin{aligned} \mathcal{A}(13) = \{&(1,2,3,6,13a_5',13a_6',13a_7',a_8,\cdots,a_{16}) \\ &(1,2,3,7,13a_5',13a_6',13a_7',a_8,\cdots,a_{16}) \\ &(1,2,4,5,13a_5',13a_6',13a_7',a_8,\cdots,a_{16}) \\ &(1,2,4,6,13a_5',13a_6',13a_7',a_8,\cdots,a_{16}) \\ &(1,2,4,7,13a_5',13a_6',13a_7',a_8,\cdots,a_{16}) \\ &(1,2,4,8,13a_5',13a_6',13a_7',a_8,\cdots,a_{16}) | a_i', a_i \in \mathbb{N}\} \subset \mathcal{A}. \end{aligned}$$

Based on the Remark 3.3.2, for each $\mathbf{a} \in \mathcal{A}(13)$, we need to take suitable $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ such that $L_{f,\beta} \otimes \mathbb{Z}_q$ are even universal for all $q \in \{2, 3, 5, 7, 11, 13\}$, yielding the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ is locally even universal. From the definition of $\mathcal{A}$, we have

$$a_5' = 1, \quad a_6' \in \{1, 2\}, \quad a_7' \in \{1, 2, 3, 4\}.$$

Hence for $\mathbf{a} \in \mathcal{A}(13)$, we may use the Proposition 3.3.1 to obtain that for any $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = 1$, $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal for all $p \in \{5, 7, 11\}$, i.e., for $\mathbf{a} \in \mathcal{A}(13)$, there is never any need to consider the local structure $L_{f,\beta} \otimes \mathbb{Z}_p$ for $p \in \{5, 7, 11\}$ in choosing $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta \in \mathbb{Z}^{10}$. In other words, it is only required to take care the local structure $L_{f,\beta} \otimes \mathbb{Z}_p$ for $p \in \{2, 3, 13\}$ in taking suitable $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta}$ is locally even universal for $\mathbf{a} \in \mathcal{A}(13)$. Moreover, one may observe that for $\mathbf{a} \in \mathcal{A}(13)$ with $(a_3, a_4, a_7') \neq (3, 6, 3)$, either there are at least 6 units(by admitting the recursion) of $\mathbb{Z}_3$ or there are 5 units and 2 primes(by admitting the recursion) of $\mathbb{Z}_3$ in $\{a_1, a_2, \cdots, a_7\}$. Therefore we may see that for any $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = 1$, $L_{f,\beta} \otimes \mathbb{Z}_3$ is (even) universal by again using the Proposition 3.3.1. Hence for the cases $(a_3, a_4, a_7') \neq (3, 6, 3)$, we need to take care only the local structure $L_{f,\beta} \otimes \mathbb{Z}_p$ for $p \in \{2, 13\}$.

1. We first consider $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1, 2, 3, 7)$.

   If $a_i \not\equiv 0 \pmod{13}$ for some $8 \leq i \leq 16$, by taking one of a_{i_8}, a_{i_8} and $a_{i_{10}}$ as an unit of $\mathbb{Z}_{13}$, we may yield that $L_{f,\beta} \otimes \mathbb{Z}_{13}$ is (even) universal for any $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = 1$ by using the Proposition 3.3.1. If $a_i \equiv 0 \pmod{13}$ for all $8 \leq i \leq 16$, then for any choice of $8 \leq i_8 < i_9 < i_{10} \leq 16$, the discriminant of $(\mathbb{Z}\alpha + \mathbb{Z}\beta, Q_f)$ is

   $$Q_f(\alpha)Q_f(\beta) - B_f(\alpha, \beta)^2 \equiv 0 \cdot 1 - 1 \not\equiv 0 \pmod{13}$$

   for $\beta := \mathbf{e}_1 \in \mathbb{Z}^{10}$. From the above discriminant argument, we may have that $\mathbb{Z}_{13}\alpha + \mathbb{Z}_{13}\mathbf{e}_1$ is unimodular. Then by using the Proposition 3.3.3, we may have that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_{13} (= (\mathbb{Z}_{13}\alpha + \mathbb{Z}_{13}\mathbf{e}_1)^{\perp})$ is isometric to

   $$\left\langle \frac{1 \cdot 2 \cdot 3}{Q_f(\alpha)Q_f(\mathbf{e}_1) - 1} \right\rangle \perp \langle 7 \rangle \perp \langle 13 \rangle \perp \langle 13a_6' \rangle \perp \cdots \perp \langle a_{i_{10}} \rangle .$$

   And then $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_{13}$ has a binary unimodular $\mathbb{Z}_{13}$-lattice(which is isometric to $\left\langle \frac{1\cdot 2\cdot 3}{Q_f(\alpha)Q_f(\mathbf{e}_1)-1} \right\rangle \perp \langle 7\rangle$) which represents every 13-adic integer of even order and a binary 13-modular $\mathbb{Z}_{13}$-lattice(which is isometric to $\langle 13\rangle \perp \langle 13a_6'\rangle$) which represents every 13-adic integer of odd order by Corollary 2.2.5. That implies that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_{13}$ is (even) universal for any $8 \le i_8 < i_9 < i_{10} \le 16$.

For $p = 2$, by using the Proposition 3.3.4 with $a_{i_{j_1}} = a_3 = 3, a_{i_{j_2}} = a_4 = 7, a_{i_{j_3}} = a_5 = 13$ and $\beta_{j_1} = \beta_{j_2} = \beta_{j_3} = 0$, we have that $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal where $\beta = \mathbf{e}_1$ for any $8 \le i_8 < i_9 < i_{10} \le 16$.

Consequently, we may conclude that for each $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1, 2, 3, 7)$, the quadratic $\mathbb{Z}$-lattice $L_{f,\mathbf{e}_1}$ is locally even universal for some $8 \le i_8 < i_9 < i_{10} \le 16$.

2. We secondly consider $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1, 2, 4, 5)$.

Through similar arguments with the case $(a_1, a_2, a_3, a_4) = (1, 2, 3, 7)$, we may assert that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_{13}$ is (even) universal by well choosing at most one extra coefficient among a_{i_8}, a_{i_9}, and $a_{i_{10}}$ in $\{a_8, \cdots, a_{16}\}$.

For $p = 2$, if $a_6' = 1$, then we may use the Proposition 3.3.4(3) with $a_{i_{j_1}} = a_4 = 5, a_{i_{j_2}} = a_5 = 13, a_{i_{j_3}} = a_6 = 13$, and $a_{i_{j_4}} = a_2 = 2$ to assert that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_2$ is even universal for any $8 \le i_8 < i_9 < i_{10} \le 16$. If $a_6' = 2$, then for any $8 \le i_8 < i_9 < i_{10} \le 16$, by using the Proposition 3.3.5 with $a_{i_{j_1}} = a_4 = 5, a_{i_{j_2}} = a_2 = 2, a_{i_{j_3}} = a_6 = 26, a_{i_{j_4}} = a_3 = 4, a_{i_{j_5}} = a_5 = 13$, we may assert that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_2$ has a $\mathbb{Z}_2$-sublattice which is isometric to

$$\langle 6\rangle \perp \langle 2\rangle \perp \langle 20\rangle \perp \langle 40\rangle$$

which is even universal over $\mathbb{Z}_2$ by Lemm 3.3.7. Therefore for $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1, 2, 4, 5)$, $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_2$ is even universal for any $8 \le i_8 < i_9 < i_{10} \le 16$.

Consequently, we may conclude that for each $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1, 2, 4, 5)$, the quadratic $\mathbb{Z}$-lattice $L_{f,\mathbf{e}_1}$ is locally even universal for some $8 \le i_8 < i_9 < i_{10} \le 16$.

3. Similarly with the above 1 and 2, one may show that for each $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1,2,4,6), (1,2,4,8)$, or $(1,2,3,6)$ and $a_7' \neq 3$, $L_{f,\mathbf{e}_1}$ is locally even universal for some $8 \leq i_8 < i_9 < i_{10} \leq 16$.

4. We now consider $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1,2,4,7)$.

   Through similar arguments with the case $(a_1, a_2, a_3, a_4) = (1,2,3,7)$ in above 1, we may yield that $L_{f,\beta} \otimes \mathbb{Z}_{13}$ is (even) universal for some $8 \leq i_8 < i_9 < i_{10} \leq 16$ where $\beta = \mathbf{e}_1 - 5\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 \in \mathbb{Z}^{10}$(note that $B_f(\alpha, \beta) = 1$).

   For $p = 2$, by using the Proposition 3.3.4(1) with $a_{i_{j_1}} = a_1 = 1, a_{i_{j_2}} = a_4 = 7, a_{i_{j_3}} = a_5 = 13$, we may yield that $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal where $\beta = \mathbf{e}_1 - 5\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 \in \mathbb{Z}^{10}$ for any $8 \leq i_8 < i_9 < i_{10} \leq 16$.

   Consequently, we may conclude that for each $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1,2,4,7)$, for $\beta := \mathbf{e}_1 - 5\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 \in \mathbb{Z}^{10}$, the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ is locally even universal for some $8 \leq i_8 < i_9 < i_{10} \leq 16$.

5. We finally consider $\mathbf{a} \in \mathcal{A}(13)$ with $(a_1, a_2, a_3, a_4) = (1,2,3,6)$ and $a_7 = 13a_7' = 13 \cdot 3$.

   Through similar arguments with the above 1, we may yield that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_{13}$ is (even) universal for some $8 \leq i_8 < i_9 < i_{10} \leq 16$.

   For $p = 3$, if there is $a_i \not\equiv 0 \pmod 3$ for some $8 \leq i \leq 16$, then by taking one of a_{i_8}, a_{i_8} and $a_{i_{10}}$ as an unit of $\mathbb{Z}_3$, we may yield that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_3$ is (even) universal for any by using the Proposition 3.3.1(2). Now assume that $a_i \equiv 0 \pmod 3$ for all $8 \leq i \leq 16$. Then for any $8 \leq i_8 < i_9 < i_{10} \leq 16$, $Q_f(\alpha) \equiv 0$ or $2 \pmod 3$ since $a_5 = 13$ and $a_6 = 13$ or 26. Therefore the discriminant of $(\mathbb{Z}\alpha + \mathbb{Z}\beta, Q_f)$ where $\beta := \mathbf{e}_1$ is

   $$Q_f(\alpha)Q_f(\beta) - B_f(\alpha, \beta) = Q_f(\alpha) \cdot 1 - 1 \not\equiv 0 \pmod 3,$$

   which implies that $(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta, Q_f)$ is unimodular. By using the Proposition 3.3.3, we may have that $L_{f,\mathbf{e}_1} \otimes \mathbb{Z}_3 (= (\mathbb{Z}_3\alpha + \mathbb{Z}_3\mathbf{e}_1)^{\perp})$ is isometric

to
$$\left\langle \frac{1\cdot 2\cdot 13}{Q_f(\alpha)Q_f(\beta)-1} \right\rangle \perp \langle 13a_6' \rangle \perp \langle 3\rangle \perp \langle 6\rangle \perp \cdots \perp \langle a_{i_{10}}\rangle .$$
Therefore $L_{f,\mathbf{e}_1}\otimes \mathbb{Z}_3$ is (even) universal for any $8\le i_8<i_9<i_{10}\le 16$ since $\left\langle \frac{1\cdot 2\cdot 13}{Q_f(\alpha)Q_f(\beta)-1} \right\rangle \perp \langle 13a_6' \rangle \perp \langle 3\rangle \perp \langle 6\rangle$ is (even) universal over $\mathbb{Z}_3$ from $a_6'\in\{1,2\}$.

For $p=2$, by using the Proposition 3.3.4(1) with $a_{i_{j_1}}=a_3=3, a_{i_{j_2}}=a_5=13, a_{i_{j_3}}=a_7=13\cdot 3$, we may see that $L_{f,\mathbf{e}_1}\otimes\mathbb{Z}_2$ is even universal for any $8\le i_8<i_9<i_{10}\le 16$.

Consequently, we may conclude that for each $\mathbf{a}\in\mathcal{A}(13)$ with $(a_1,a_2,a_3,a_4)=(1,2,3,6)$ and $a_7=13a_7'=13\cdot 3$, $L_{f,\mathbf{e}_1}$ is locally even universal for some $8\le i_8<i_9<i_{10}\le 16$.

Case II. $\mathcal{A}(\mathbf{11})$

One may yield that for each $\mathbf{a}\in\mathcal{A}(11)$, there are suitable $8\le i_8<i_9<i_{10}\le 16$ and $\beta\in\mathbb{Z}^{10}$ for which the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ is locally even universal through similar processings with the **Case I**. $\mathcal{A}(\mathbf{13})$ and that are omitted here.

Case III. $\mathcal{A}(\mathbf{7})$

For any $\mathbf{a}\in\mathcal{A}(7)$, $\mathbf{a}$ satisfies one of the following three conditions

(1) There are five units and one prime of $\mathbb{Z}_7$ in $\{a_1,\cdots,a_6\}$ by admitting recursion and $a_7=49$.

(2) There are only four units of $\mathbb{Z}_7$ in $\{a_1,\cdots,a_7\}$ by admitting recursion.

(3) There are only three units of $\mathbb{Z}_7$ in $\{a_1,\cdots,a_7\}$ by admitting recursion.

First, we consider $\mathbf{a}\in\mathcal{A}(7)$ satisfying the condition (3), which are forms of

$$(1,2,3,7a_4',\cdots,7a_7',a_8,\cdots,a_{16}) \text{ or } (1,2,4,7a_4',\cdots,7a_7',a_8,\cdots,a_{16})\in\mathcal{A}.$$

Based on the Remark 3.3.2, for each $\mathbf{a}\in\mathcal{A}(7)\setminus\mathcal{A}(13)\cup\mathcal{A}(11)$, we need to take suitable $8\le i_8<i_9<i_{10}\le 16$ and $\beta\in\mathbb{Z}^{10}$ for which $L_{f,\beta}\otimes\mathbb{Z}_q$ are

even universal for all $q \in \{2,3,5,7\}$, yielding the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ is locally even universal. From the definition of $\mathcal{A}$, we have $a'_4 = 1, a'_5 \in \{1,2\}$, and $a'_6 \in \{1,2,3,4\}$. Hence for $\mathbf{a} \in \mathcal{A}(7)$ satisfying (3), we may use the Proposition 3.3.1 to obtain that for any $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha,\beta) = 1$, $L_{f,\beta} \otimes \mathbb{Z}_5$ is (even) universal, i.e., for $\mathbf{a} \in \mathcal{A}(7)$, there is never any need to consider the local structure $L_{f,\beta} \otimes \mathbb{Z}_5$ in choosing suitable $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$. In other words, it is only required to take care the local structure $L_{f,\beta} \otimes \mathbb{Z}_p$ for $p \in \{2,3,7\}$ when we choose suitable $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta}$ is locally even universal for $\mathbf{a} \in \mathcal{A}(7)$. Moreover, for any $\mathbf{a} \in \mathcal{A}(7)$ satisfying (3) except the cases $(a_3, a'_6, a'_7) = (3,3,3)$ or $(3,3,6)$, either there are at least six units(by admitting the recursion) of $\mathbb{Z}_3$ or there are five units and two primes(by admitting the recursion) of $\mathbb{Z}_3$ in $\{a_1, a_2, \cdots, a_7\}$. Therefore we may have that for any $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha,\beta) = 1$, $L_{f,\beta} \otimes \mathbb{Z}_3$ is (even) universal by again using the Proposition 3.3.1. Hence for the cases $(a_3, a'_6, a'_7) \neq (3,3,3)$ and $(3,3,6)$, we need to take care only the local structure $L_{f,\beta} \otimes \mathbb{Z}_p$ for $p \in \{2,7\}$.

1. First consider $\mathbf{a} \in \mathcal{A}(7)$ satisfying (3) with $(a_3, a'_6, a'_7) \neq (3,3,3), (3,3,6)$.

   From the definition of $\mathcal{A}$, we have

   $$(a'_4, a'_5, a'_6) \in \{(1,1,1), (1,1,2), (1,1,3), (1,2,2), (1,2,3), (1,2,4)\}.$$

   If $(a'_4, a'_5, a'_6) = (1,1,1)$, then for $\beta := -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + 3\mathbf{e}_6 \in \mathbb{Z}^{10}$, we may use the Proposition 3.3.5(with $a_{i_{j_1}} = a_1 = 1, a_{i_{j_2}} = a_2 = 2, a_{i_{j_3}} = a_3 = 3, a_{i_{j_4}} = a_4 = 7, a_{i_{j_5}} = a_5 = 7$) to yield that $L_{f,\beta} \otimes \mathbb{Z}_7$ contains an (even) universal $\mathbb{Z}_7$-sublattice which is isometric to

   $$\langle 2 \cdot 1 \cdot 3 \rangle \perp \langle 3 \cdot 3 \cdot 6 \rangle \perp \langle 7 \cdot 6 \cdot 13 \rangle \perp \langle 7 \cdot 13 \cdot 20 \rangle$$

   for any $8 \le i_8 < i_9 < i_{10} \le 16$, which implies $L_{f,\beta} \otimes \mathbb{Z}_7$ is (even) universal. On the other hand, for $p = 2$, we may use the Proposition 3.3.4(1)(with $a_{i_{j_1}} = a_1 = 1, a_{i_{j_2}} = a_3 = 3, a_{i_{j_3}} = a_4 = 7$) to yield that $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal for any $8 \le i_8 < i_9 < i_{10} \le 16$. Consequently, we conclude that for each $\mathbf{a} \in \mathcal{A}(7)$ satisfying (3) with $(a_3, a'_6, a'_7) \neq$

$(3,3,3),(3,3,6)$ and $(a_4',a_5',a_6')=(1,1,1)$, the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ is locally even universal where $\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5+3\mathbf{e}_6\in\mathbb{Z}^{10}$ and $i_8=8,i_9=9,i_{10}=10$.

For the remaining cases $\mathbf{a}\in\mathcal{A}(7)$ satisfying (3) with $(a_3,a_6',a_7')\neq(3,3,3)$, $(3,3,6)$ and $(a_4',a_5',a_6')=(1,1,2),(1,1,3)$, $(1,2,2),(1,2,3)$, or $(1,2,4)$. One may see that $L_{f,\beta}$ is locally even universal and $i_8=8,a_9=9,i_{10}=10$ for the below $\beta\in\mathbb{Z}^{10}$ case by case through similar processings with the above. That is omitted here.

(1) $(a_4',a_5',a_6')=(1,1,1);\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5+3\mathbf{e}_6$.

(2) $(a_4',a_5',a_6')=(1,1,2);\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4+4\mathbf{e}_5-\mathbf{e}_6$.

(3) $(a_4',a_5',a_6')=(1,1,3);\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5+\mathbf{e}_6$.

(4) $(a_4',a_5',a_6')=(1,2,2);\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5+2\mathbf{e}_6$.

(5) $(a_4',a_5',a_6')=(1,2,3);\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+6\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6$.

(6) $(a_4',a_5',a_6')=(1,2,4);\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5+\mathbf{e}_6$.

2. Consider $\mathbf{a}\in\mathcal{A}(7)$ satisfying (3) with $(a_3,a_6',a_7')=(3,3,3)$ or $(3,3,6)$.

For $p=7$, we may see that $L_{f,\beta}\otimes\mathbb{Z}_7$ is (even) universal for $\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+6\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6\in\mathbb{Z}^{10}$ and any $8\le i_8<i_9<i_{10}\le 16$ through similar arguments with the above.

For $p=3$, if $a_i\not\equiv 0\pmod 3$ for some $8\le i\le 16$, by taking one of a_{i_8},a_{i_9} and $a_{i_{10}}$ as an unit of $\mathbb{Z}_3$, we may yield that $L_{f,\beta}\otimes\mathbb{Z}_3$ is (even) universal for any $\beta\in\mathbb{Z}^{10}$ by using the Proposition 3.3.1. If $a_i\equiv 0\pmod 3$ for all $8\le i\le 16$, one may easily observe that the discriminant of $(\mathbb{Z}\alpha+\mathbb{Z}\beta,Q_f)$ is

$$Q_f(\alpha)Q_f(\beta)-B_f(\alpha,\beta)^2\equiv Q_f(\alpha)\cdot 1-1\not\equiv 0\pmod 3$$

where $\beta=-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+6\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6\in\mathbb{Z}^{10}$ and $i_8=8,a_9=9,i_{10}=10$ since $Q_f(\alpha)\equiv 0$ or $2\pmod 3$. From the above discriminant argument, we have that $(\mathbb{Z}_3\alpha+\mathbb{Z}_3\beta,Q_f)$ is unimodular. By using the Proposition 3.3.3, we may have that $L_{f,\beta}\otimes\mathbb{Z}_3(=(\mathbb{Z}_3\alpha+\mathbb{Z}_3\beta)^\perp)$ is isometric to an (even) universal $\mathbb{Z}_3$-lattice

$$\left\langle \frac{1\cdot 2\cdot 7}{Q_f(\alpha)Q_f(\beta)-1}\right\rangle \perp \langle 7a_5'\rangle \perp \langle 3(=a_3)\rangle \perp \langle 3a_6'\rangle \perp \cdots \perp \langle a_{i_{10}}\rangle .$$

Lastly, for $p=2$, we may use the Proposition 3.3.4(1)(with $a_{i_{j_1}} = a_1 = 1, a_{i_{j_2}} = a_3 = 3, a_{i_{j_3}} = a_6 = 7\cdot 3$) to assert that $L_{f,\beta}\otimes\mathbb{Z}_2$ is even universal for $\beta = -\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+6\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6\in\mathbb{Z}^{10}$ and any $8\le i_8<i_9<i_{10}\le 16$.

Consequently, we conclude that for each $\mathbf{a}\in\mathcal{A}(7)$ satisfying (3) with $(a_3,a_6',a_7') = (3,3,3)$ or $(3,3,6)$, $L_{f,\beta}$ is locally even universal where $\beta = -\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+6\mathbf{e}_4-\mathbf{e}_5-\mathbf{e}_6\in\mathbb{Z}^{10}$ for some $8\le i_8<i_9<i_{10}\le 16$.

3. Next consider $\mathbf{a}\in\mathcal{A}(7)$ satisfying (3) with $a_3=4$.

   One may use the Proposition 3.3.1 to obtain that $L_{f,\beta}\otimes\mathbb{Z}_q$ is (even) universal for any $8\le i_8<i_9<i_{10}\le 16$ and $\beta\in\mathbb{Z}^{10}$ for all odd primes $q\ne 7$.

   For $p=7$, if there is $a_i\not\equiv 0 \pmod 7$ for some $8\le i\le 16$ we take one of a_{i_8}, a_{i_8} and $a_{i_{10}}$ as an unit of $\mathbb{Z}_7$. By reordering if it is necessary, we assume that $a_{i_8}\not\equiv 0 \pmod 7$. Put $\beta^{(1)} := \mathbf{e}_1$ and $\beta^{(2)} := -\mathbf{e}_1+\mathbf{e}_2$ in $\mathbb{Z}^{10}$. One may see that by considering the discriminants of $(\mathbb{Z}_7\alpha+\mathbb{Z}_7\beta^{(1)},Q_f)$ and $(\mathbb{Z}_7\alpha+\mathbb{Z}_7\beta^{(2)},Q_f)$, since $Q_f(\beta^{(1)})=1\not\equiv Q_f(\beta^{(2)})=3 \pmod 7$, at least one of $(\mathbb{Z}_7\alpha+\mathbb{Z}_7\beta^{(1)},Q_f)$ and $(\mathbb{Z}_7\alpha+\mathbb{Z}_7\beta^{(2)},Q_f)$ is unimodular. For an unimodular $(\mathbb{Z}_7\alpha+\mathbb{Z}_7\beta^{(k)},Q_f)$, we may use the Proposition 3.3.3 to yield that $L_{f,\beta^{(k)}}\otimes\mathbb{Z}_7$ is (even) unimodular.

   For $p=2$, for each candidate $(a_5',a_6')\in\{(1,1),(1,2),(1,3),(2,2),(2,3),(2,4)\}$ we can check that $L_{f,\beta^{(1)}}\otimes\mathbb{Z}_2$ is even universal for any $8\le i_8<i_9<i_{10}\le 16$ by using the Proposition 3.3.4 and Proposition 3.3.5 with well reordered $j_1,\cdots,j_5\in\{2,3,\cdots,6\}$.
   On the other hand, when $(a_5',a_6')=(1,1)$ or $(1,3)$, $L_{f,\beta^{(2)}}\otimes\mathbb{Z}_2$ is even universal by Proposition 3.3.6(1). When $(a_5',a_6')=(2,2)$ or $(2,4)$, by using the Proposition 3.3.6 (2)(with $a_{i_{j_1}}=a_4=7, a_{i_{j_2}}=a_5=14, a_{i_{j_3}}=a_6=7a_6', a_{i_{j_4}}=a_3=4$), we may see that $L_{f,\beta^{(2)}}\otimes\mathbb{Z}_2$ contains an even

universal $\mathbb{Z}_2$-sublattice

$$\langle b_0\rangle \perp \langle b_1\rangle \perp \langle b_2\rangle \perp \langle b_3\rangle$$

with $(\nu_2(b_0),\nu_2(b_1),\nu_2(b_2),\nu_2(b_3))=(1,1,1,2)$ or $(1,1,2,2)\in\Delta_2$.
When $(a_5',a_6')=(2,3)$, by using the Proposition 3.3.6 (2)(with $a_{i_{j_1}}=a_4=7, a_{i_{j_2}}=a_5=14, a_{i_{j_3}}=a_3=4, a_{i_{j_4}}=a_6=21$), we may see that $L_{f,\beta^{(2)}}\otimes\mathbb{Z}_2$ contains an even universal $\mathbb{Z}_2$-sublattice

$$\langle b_0\rangle \perp \langle b_1\rangle \perp \langle b_2\rangle \perp \langle b_3\rangle$$

with $(\nu_2(b_0),\nu_2(b_1),\nu_2(b_2),\nu_2(b_3))=(1,1,2,1)$.
When $(a_5',a_6')=(1,2)$, we need extra help of $a_7=7a_7'$. Note that $a_7'=2,3,4,$ or 5. When $a_7'=3$ or 5, we may again use the Lemma 3.3.6 (1) to show that $L_{f,\beta}\otimes\mathbb{Z}_2$ is even universal. When $a_7'=2$ or 4, by using the Proposition 3.3.6 (2)(with $a_{i_{j_1}}=a_4=7, a_{i_{j_2}}=a_6=14, a_{i_{j_3}}=a_7=7a_7', a_{i_{j_4}}=a_3=4$), we may yield that $L_{f,\beta}\otimes\mathbb{Z}_2$ contains an even universal $\mathbb{Z}_2$-sublattice

$$\langle b_0\rangle \perp \langle b_1\rangle \perp \langle b_2\rangle \perp \langle b_3\rangle$$

with $(\nu_2(b_0),\nu_2(b_1),\nu_2(b_2),\nu_2(b_3))=(1,1,1,2)$ or $(1,1,2,2)\in\Delta_2$.
Consequently, for $\mathbf{a}\in\mathcal{A}(7)$ satisfying (3) with $a_3=4$, both of

$$L_{f,\beta^{(1)}}\otimes\mathbb{Z}_2 \text{ and } L_{f,\beta^{(2)}}\otimes\mathbb{Z}_2$$

are even universal for any $8\le i_8<i_9<i_{10}\le 16$.

As a result, $L_{f,\beta}$ is locally even universal for some $8\le i_8<i_9<i_{10}\le 16$ and $\beta\in\{\mathbf{e}_1,-\mathbf{e}_1+\mathbf{e}_2\}$.

On the other hand, for $\mathbf{a}\in\mathcal{A}(7)$ satisfying (3) with $a_3=4$, in the cases $a_i\equiv 0 \pmod 7$ for all $8\le i\le 16$, there is serious issue which is $L_{f,\beta}$ is never locally even universal for any $8\le i_8<i_9<i_{10}\le 16$ and $\beta\in\mathbb{Z}^{10}$. More precisely, $L_{f,\beta}\otimes\mathbb{Z}_7$ is never (even) universal for any $8\le i_8<i_9<i_{10}\le 16$ and $\beta\in\mathbb{Z}^{10}$. It is treated in Section 3.4 and Section 3.5.

For the case $\mathbf{a} \in \mathcal{A}(7)$ satisfying the above conditions (1) or (2), one may take $\beta \in \mathbb{Z}^{10}$ and $8 \le i_8 < i_9 < i_{10} \le 16$ for which $L_{f,\beta}$ is locally even universal through similar processings with the **Case I**. $\mathcal{A}(\mathbf{13})$ and the above argument in this **Case III**. $\mathcal{A}(\mathbf{7})$.

Case IV. $\mathcal{A}(\mathbf{5})$

For $\mathbf{a} \in \mathcal{A}(5)$, $\mathbf{a}$ satisfies one of the following three conditions

(1) There are five units and at most one prime of $\mathbb{Z}_5$ in $\{a_1, \cdots, a_7\}$ by admitting recursion.

(2) There are only four units of $\mathbb{Z}_5$ in $\{a_1, \cdots, a_7\}$ by admitting recursion.

(3) There are only three units of $\mathbb{Z}_5$ in $\{a_1, \cdots, a_7\}$ by admitting recursion.

Based on the Remark 3.3.2, for each $\mathbf{a} \in \mathcal{A}(5) \setminus \mathcal{A}(13) \cup \mathcal{A}(11) \cup \mathcal{A}(7)$, we need to take suitable $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta} \otimes \mathbb{Z}_q$ are even universal for all $q \in \{2, 3, 5\}$, yielding the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ is locally even universal.

Through similar processings with the **Case I**. $\mathcal{A}(\mathbf{13})$ and **Case III**. $\mathcal{A}(\mathbf{7})$ by using the Proposition 3.3.1, Proposition 3.3.3, Proposition 3.3.4, Proposition 3.3.5 and Proposition 3.3.6, one may check that $L_{f,\beta}$ is locally even universal for some $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ for each $\mathbf{a} \in \mathcal{A}$ except

$$\mathbf{a} = (1, 1, 3, 6, 10, 15a'_6, 15a'_7, 3a'_8 \cdots, 3a'_{16}) \text{ in the case (2),} \tag{3.3.6}$$

$$\mathbf{a} = (1, 1, 3, 5a'_4, \cdots, 5a'_{16}) \text{ and } \mathbf{a} = (1, 2, 2, 5a'_4, \cdots, 5a'_{16}) \text{ in the case (3).} \tag{3.3.7}$$

The above $\mathbf{a} \in \mathcal{A}(5)$ have also serious issue which is $L_{f,\beta}$ is never locally even universal for any $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$. Precisely, in the case (3.3.6), $L_{f,\beta} \otimes \mathbb{Z}_3$ is never (even) universal and in the case (3.3.7), $L_{f,\beta} \otimes \mathbb{Z}_5$ is never (even) universal. It is treated in next section.

Case V. $\mathcal{A}(\mathbf{3})$

For $\mathbf{a} \in \mathcal{A}(3)$, $\mathbf{a}$ satisfies one of the following four conditions

(1) There are five units and at most one prime of $\mathbb{Z}_3$ in $\{a_1, \cdots, a_7\}$ by admitting recursion.

(2) There are only four units of $\mathbb{Z}_3$ in $\{a_1, \cdots, a_7\}$ by admitting recursion.

(3) There are only three units of $\mathbb{Z}_3$ in $\{a_1, \cdots, a_7\}$ by admitting recursion.

(4) There are only two units of $\mathbb{Z}_3$ in $\{a_1, \cdots, a_7\}$ by admitting recursion.

Based on the Remark 3.3.2, for each $\mathbf{a} \in \mathcal{A}(3) \setminus \bigcup\limits_{p \in S} \mathcal{A}(p)$ where $S = \{5, 7, 11, 13\}$, we want to find suitable $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta} \otimes \mathbb{Z}_q$ are even universal for all $q \in \{2, 3\}$, yielding that the quadratic $\mathbb{Z}$-lattice $L_{f,\beta}$ is locally even universal.

Going through a similar process as in **Case I**. $\mathcal{A}(\mathbf{13})$ and **Case III**. $\mathcal{A}(\mathbf{7})$ by using Proposition 3.3.1, Proposition 3.3.3, Proposition 3.3.4, Proposition 3.3.5 and Proposition 3.3.6 for each $\mathbf{a} \in \mathcal{A}'(3)$, one may take $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta}$ is locally even universal except

$$(1, 2, 3, 6, 8, 8a'_6, 24, 8a'_8, \cdots, 8a'_{16}) \text{ in the case (2),} \tag{3.3.8}$$

$$(1, 1, 3a'_3, \cdots, 3a'_k, 3n+1, 3a'_{k+2}, \cdots, 3a'_{16}) \text{ in the case (3) or (4),} \tag{3.3.9}$$

$$(1, 1, 3a'_3, \cdots, 3a'_{16}) \text{ and } (1, 2, 3a'_3, \cdots, 3a'_{16}) \text{ in the case (4)} \tag{3.3.10}$$

where $3 \le k+1 \le 16$. We postpone the treatment of these exceptional vectors $\mathbf{a}$ listed in $\mathcal{A}'(3)$ of Table 3.1 to Section 3.4 and Section 3.5 where it is turned out that $L_{f,\beta}$ is never locally even universal for any choice of $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$. I might add in advance that in the case (3.3.8), $L_{f,\beta} \otimes \mathbb{Z}_2$ is never even universal and in the case (3.3.9) and (3.3.10), $L_{f,\beta} \otimes \mathbb{Z}_3$ is never (even) universal.

Case VI. $\mathcal{A}(\mathbf{2})$

Let

$$\mathcal{A}(2) := \mathcal{A} \setminus \bigcup_{p \in S} \mathcal{A}(p) \text{ where } S = \{3, 5, 7, 11, 13\}.$$

Based on the Remark 3.3.2, finally for each $\mathbf{a} \in \mathcal{A}(2)$ it is required to check whether there are suitable $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ for which $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal.

Going through a similar process with the **Case I**. $\mathcal{A}(\mathbf{13})$ and **Case III**. $\mathcal{A}(\mathbf{7})$ by using the Proposition 3.3.4, Proposition 3.3.5 and Proposition 3.3.6, one

Table 3.1: Dropouts

$\mathcal{A}'(7)$	$(1,2,4,7a'_4,\cdots,7a'_{16})$
$\mathcal{A}'(5)$	$(1,1,3,6,10,15a'_6,15a'_7,3a'_8\cdots,3a'_{16})$, $(1,1,3,5a'_4,\cdots,5a'_{16}),(1,2,2,5a'_4,\cdots,5a'_{16})$
$\mathcal{A}'(3)$	$(1,2,3,6,8,8a'_6,24,8a'_8,\cdots,8a'_{16})$, $(1,1,3a'_3,\cdots,3a'_{16}),(1,2,3a'_3,\cdots,3a'_{16})$, $(1,1,3a'_3,\cdots,3a'_k,3n+1,3a'_{k+2},\cdots,3a'_{16})$ for some $3\le k+1\le 16$,
$\mathcal{A}'(2)$	$(1,2,2,5,8a'_5,\cdots,8a'_{16}),(1,2,2,5,10,16a'_5,\cdots,16a'_{16})$, $(1,2,3,6,8a'_5,\cdots,8a'_{16}),(1,2,4,4,7,16a'_5,\cdots,16a'_{16})$, $(1,2,4,4,9,16a'_5,\cdots,16a'_{16}),(1,2,4,4,11,16a'_5,\cdots,16a'_{16})$, $(1,2,4,5,12,16a'_5,\cdots,16a'_{16}),(1,2,4,7,12,16a'_5,\cdots,16a'_{16})$, $(1,2,2,3,8a'_5,\cdots,8a'_{16}),(1,2,4,4,5,16a'_6,\cdots,16a'_{16})$

may take suitable $8\le i_8<i_9<i_{10}\le 16$ and $\beta\in\mathbb{Z}^{10}$ for which $L_{f,\beta}\otimes\mathbb{Z}_2$ is even universal for each $\mathbf{a}\in\mathcal{A}(2)$ except the elements in $\mathcal{A}'(2)$ of Table 3.1.

To finish this lengthy calculations, some computers are used.
And the remaining dropouts in Table 3.1 are treated in the remaining of this chapter.

3.4 Generalization

As a first step to clean up the dropouts in Table 3.1, we suggest an improvement of Lemma 3.2.5. Under the same assumptions and notations in Lemma 3.2.5, for each $0\le t_1\le s+1$ and $-1\le r_1\le 2^{16}-1$, suppose that there is $\beta(t_1,r_1)\in\mathbb{Z}^{10}$ with $B_f(\alpha,\beta(t_1,r_1))=1$ for which

$$Q_f(\mathbf{x}(t_1,r_1))+r_1^2Q_f(\beta(t_1,r_1))-r_1\equiv 2t_1 \pmod{2s}$$

for some $\mathbf{x}(t_1, r_1) \in L_{f,\beta(t_1,r_1)}$. Then the m-gonal form $f_m(\mathbf{x})$ may represent all the $t_1(m-2)+r_1$ modulo $s(m-2)$ in the interval

$$[0, (\frac{1}{2}\max_{(t_1,r_1)}\{Q_f(\mathbf{x}(t_1,r_1)) + r_1^2 Q_f(\beta(t_1,r_1)) - r_1\} + 1)(m-2)]$$

because

$$f_m(\mathbf{x}(t_1,r_1)+r_1\beta(t_1,r_1)) = \frac{m-2}{2}\{Q_f(\mathbf{x}(t_1,r_1)) + r_1^2 Q_f(\beta(t_1,r_1)) - r_1\} + r_1.$$

Actually, in the Lemma 3.2.5, the sole vector $\beta \in \mathbb{Z}^{10}$ fulfils the duties for $\beta(t_1,r_1) \in \mathbb{Z}^{10}$ for all $0 \le t_1 \le s+1$ and $-1 \le r_1 \le 2^{16}-1$.

In the following lemma, as a kind of generalization of the Lemma 3.2.5, we show a sufficient local condition for the existence of such above $\beta(t_1,r_1) \in \mathbb{Z}^{10}$ for all $0 \le t_1 \le s+1$ and $-1 \le r_1 \le 2^{16}-1$.

Lemma 3.4.1. *For $m \ge 2^{16}+3$, let an m-gonal form*

$$F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$$

represent every positive integer up to $m-4$. For a prime p_0, if there are a 10-subtuple $(a_{i_1}, \cdots, a_{i_{10}})$ of $(a_1, \cdots, a_{16})$ and N vectors $\beta^{(k)} \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta^{(k)}) = 1$ for all $1 \le k \le N$ where $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$ satisfying the following two local conditions,

$$L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ are even universal for all primes } p \ne p_0 \tag{3.4.1}$$

for each $1 \le k \le N$ and

$$\bigcup_{k=1}^{N}\{Q_f(\mathbf{x}) + r_1^2 \cdot Q_f(\beta^{(k)})^2 - r_1 | \mathbf{x} \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_{p_0}\} = 2\mathbb{Z}_{p_0} \tag{3.4.2}$$

for each $r_1 \in \mathbb{Z}_{p_0}$, then we may have a constant

$$C_{a_1,\cdots,a_{16}} > 0$$

which is dependent only on the first 16 coefficients of $F_m(\mathbf{x})$ from a_1 to a_{16} such that $F_m(\mathbf{x})$ represents all positive integers greater than or equal to $C_{a_1,\cdots,a_{16}}(m-2)$ regardless of the rear part $\sum\limits_{i=17}^{n} a_iP_m(x_i)$.

Proof. One may show that for each $0 \le t_1 \le s+1$ and $-1 \le r_1 \le 2^{16}-1$, there is $\beta(t_1,r_1) \in \{\beta^{(1)},\cdots,\beta^{(N)}\}$ for which

$$Q_f(\mathbf{x}) + r_1^2 Q_f(\beta(t_1,r_1)) - r_1 \equiv 2t_1 \pmod{2s}$$

is solvable on $L_{f,\beta(t_1,r_1)}$ by using the Theorem 2.3.3. And then we may finally yield that for each $0 \le t_1 \le s+1$ and $-1 \le r_1 \le 2^{16}-1$, we may get $\mathbf{x} + r_1\beta(t_1,r_1) \in \mathbb{Z}^{10}$ for which

$$\begin{cases} f_m(\mathbf{x} + r_1\beta(t_1,r_1)) \equiv t_1(m-2) + r_1 \pmod{s(m-2)} \\ 0 \le f_m(\mathbf{x} + r_1\beta(t_1,r_1)) \le (C_{\mathbf{a}}-1)(m-2) \end{cases}$$

through a similar argument with the proof of Lemma 3.2.5 where

$$C_{\mathbf{a}} := \frac{1}{2}\max_{(t_1,r_1)}\{Q_f(\mathbf{x}(t_1,r_1)) + r_1^2 Q_f(\beta(t_1,r_1)) - r_1\} + 1.$$

□

But the problem of the dropouts in Table 3.1 is for any its 10-subtuple $(a_{i_1},\cdots,a_{i_{10}})$ with $1 \le i_1 < i_2 < \cdots < i_{10} \le 16$, there is (t_1,r_1) for which there is no such the above $\beta(t_1,r_1) \in \mathbb{Z}^{10}$. There is a reason for it. From now on, we overcome the problem.

Henceforth, in this section, we consider for $m \ge 2^{17}s$, m-gonal forms having its first 16 coefficients as one of an 16-tuple in Table 3.1 which represent every positive integer up to $s(m-2)(> m-4)$. A sum of 6 components among the first 16 components of $F_m(\mathbf{x})$ are likewise used to represent all the multiples of $s(m-2)$. In the previous sections, we showed that for having first 16 coefficients not in Table 3.1 any m-gonal form which is escalated to represent every positive integer up to $m-4$, its carefully choosen remaining part except some 6 componets which are used to represent all the multiples of $s(m-2)$ may represent every residue modulo $s(m-2)$ in an interval

$[0, C(m-2)]$ where C is a constant. Still our goal is to show that the remaining part represents complete residues modulo $s(m-2)$ in the interval $[0, C(m-2)]$ for the m-gonal form $F_m(\mathbf{x})$ having its first coefficients in the Table 3.1 and escalated to represent every positive integer up to $s(m-2)$.

By our assumption, for any representative $0 \le t(m-2)+r < s(m-2)$ modulo $s(m-2)$, there may be a vector $\mathbf{x}(t,r) = (x_1(t,r), \cdots, x_n(t,r)) \in \mathbb{Z}^n$ which represents $t(m-2)+r$, i.e., $F_m(\mathbf{x}(t,r)) = t(m-2)+r$. On the other hand, some 6 components among the first 16 components of $F_m(\mathbf{x})$ are used to represent all the multiples of $s(m-2)$. So we desire that the remaining 10 components among the first 16 components handle the absent of the 6 components.

For each $0 \le t < s$ and $0 \le r < m-2$, we write

$$\sum_{i=1}^{16} a_i P_m(x_i(t,r)) =: t_1(m-2) + r_1 \quad \text{with } [-\frac{m}{2}] \le r_1 \le [\frac{m-2}{2}].$$

Since $\sum_{i=1}^{16} a_i P_m(x_i(t,r)) < s(m-2)$, $|x_i(t,r)| < s$ for all $1 \le i \le 16$. In fact, the absolute values are may much smaller than s. However, in here our point is just that the possible $x_i(t,r)$'s are finitely many independently on m. Note that from $[-\frac{m}{2}] \le r_1 \le [\frac{m-2}{2}]$, we have exactly

$$r_1 = \sum_{i=1}^{16} a_i \cdot x_i(t,r)$$

since we may have that $|r_1| \le \sum_{i=1}^{16} a_i \cdot |x_i(t,r)| < s(a_1 + \cdots + a_{16}) \le s(2^{16}-1)$. Thus regardless of m, t_1 and r_1 always satisfy the followings

$$\begin{cases} 0 \le t_1 \le s+1 \\ |r_1| < s(2^{16}-1). \end{cases} \tag{3.4.3}$$

For the convenience of notation, we define a set

$$S_{f+f^*,r_1} := \{t' \in \mathbb{Z} | t'(m-2)+r_1 = \sum_{i=1}^{16} a_i P_m(x_i) \le s(m-2) \text{ for some } x_i \in \mathbb{Z}\}$$

by the set of all the possible t_1's for each $[-\frac{m}{2}] \le r_1 \le [\frac{m-2}{2}]$ where $(f_m + f_m^*)(\mathbf{x}) = \sum_{i=1}^{16} a_i P_m(x_i)$ and an equation

$$Q_f(\mathbf{x}) := Q_f(\mathbf{x}) + r_1^2 \cdot Q_f(\beta^{(k)})^2 - r_1.$$

Then by (3.4.3), we may have that

$$\begin{cases} S_{f+f^*,r_1} \neq \emptyset \text{ only if } |r_1| \le s(2^{16}-1) \\ S_{f+f^*,r_1} \subseteq \{n \in \mathbb{Z} | 0 \le n \le s+1\}. \end{cases}$$

Now our goal is to show that even though the carefully choosen m-gonal subform $f_m(\mathbf{x})$ of rank 10 does not exactly represent the value $t_1(m-2)+r_1$ where $t_1 \in S_{f+f^*,r_1}$ and $|r_1| < s(2^{16}-1)$, it may represent an integer which is equivalent with $t_1(m-2)+r_1$ modulo $s(m-2)$ in an interval $[0,(C-s)m]$ by showing that there is $\beta(t_1,r_1) \in \mathbb{Z}^{10}$ for all $t_1 \in S_{f+f^*,r_1}$ and $|r_1| < s(2^{16}-1)$. And then we may obtain

$$f_m(\mathbf{y}(t_1,r_1)) + \sum_{i=17}^{n} a_i P_m(x_i(t,r)) \equiv t(m-2)+r \pmod{s(m-2)}$$

for $f_m(\mathbf{y}(t_1,r_1)) \equiv t_1(m-2)+r_1 \pmod{s(m-2)}$ with $0 \le f_m(\mathbf{y}(t_1,r_1)) \le (C-s)(m-2)$ yielding $f_m(\mathbf{y}(t_1,r_1)) + \sum_{i=17}^{n} a_i P_m(x_i(t,r)) \le C(m-2)$, then we are done.

In the below Lemma 3.4.2, we show a sufficient local condition for the existence of such the above $\beta(t_1,r_1) \in \mathbb{Z}^{10}$ for all $t_1 \in S_{f+f^*,r_1}$ and $|r_1| < s(2^{16}-1)$.

Lemma 3.4.2. *For $m \ge 2^{17}s$, let an m-gonal form $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$ represent all positive integers up to $s(m-2)$. If there are a prime p_0, 10-*

tuple $(i_1, \cdots, i_{10})$ satisfying $1 \le i_1 < \cdots < i_{10} \le 16$ and N vectors $\beta^{(k)} \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta^{(k)}) = 1$ for all $1 \le k \le N$ where $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$ satisfying the following two conditions,

$$L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ are even universal for all primes } p \neq p_0, \tag{3.4.4}$$

for each $1 \le k \le N$ and

$$\bigcup_{k=1}^{N} \{Q_f(\mathbf{x}) | \mathbf{x} \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_{p_0}\} \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_{p_0} \tag{3.4.5}$$

for each $r_1 \in \mathbb{Z}_{p_0}$, then there is

$$C_{\mathbf{a}} := C_{a_1,\cdots,a_{16}} > 0$$

such that $F_m(\mathbf{x})$ represents every positive integer greater than or equal to $C_{\mathbf{a}}(m-2)$ regardless of the rear part $\sum\limits_{i=17}^{n} a_iP_m(x_i)$.

Proof. For each representative $0 \le t(m-2) + r < s(m-2)$, take a vector $\mathbf{x}(t,r) = (x_1(t,r), \cdots, x_n(t,r)) \in \mathbb{Z}^n$ such that

$$F_m(\mathbf{x}(t,r)) = t(m-2) + r.$$

We write

$$\sum_{i=1}^{16} a_iP_m(x_i(t,r)) =: t_1(m-2) + r_1$$

where $[-\frac{m}{2}] \le r_1 \le [\frac{m-2}{2}]$. Since $t_1 \in S_{f+f^*,r_1}$, we may take a $\beta(t_1,r_1) \in \{\beta^{(1)}, \cdots, \beta^{(N)}\}$ such that

$$2t_1 \in \{Q_f(\mathbf{x}) + r_1^2 \cdot Q_f(\beta(t_1,r_1))^2 - r_1 | \mathbf{x} \in L_{f,\beta(t_1,r_1)} \otimes \mathbb{Z}_{p_0}\}$$

by our assumption. On the other hand, since $L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p$ are even universal for all primes p except p_0, we may conclude that the quadratic $\mathbb{Z}$-lattice $(L_{f,\beta(t_1,r_1)}, Q_f)$ locally represents $2t_1 - r_1^2 \cdot Q_f(\beta(t_1,r_1))^2 + r_1$. So by using the Theorem 2.3.3, we may obtain that $(L_{f,\beta(t_1,r_1)}, Q_f)$ represents an integer which is equivalent with $2t_1 - r_1^2 \cdot Q_f(\beta(t_1,r_1))^2 + r_1$ modulo $2s$. Take

$\mathbf{y}'(t_1,r_1)\in L_{f,\beta(t_1,r_1)}$ for which

$$Q_f(\mathbf{y}'(t_1,r_1))\equiv 2t_1-r_1^2\cdot Q_f(\beta(t_1,r_1))^2+r_1 \pmod{2s}.$$

Then since

$$f_m(\mathbf{y}'(t_1,r_1)+r_1\beta(t_1,r_1))\equiv t_1(m-2)+r_1 \pmod{s(m-2)},$$

we have that

$$f_m(\mathbf{y}'(t_1,r_1)+r_1\beta(t_1,r_1))+\sum_{i=17}^{n}a_iP_m(x_i(t,r))\equiv t(m-2)+r \pmod{s(m-2)}.$$

Thus we may get that the m-gonal form $F_m(\mathbf{x})$ without some 6 components among the first 16 components represents complete residues modulo $s(m-2)$. Finally putting

$$C_{\mathbf{a}}:=\frac{1}{2}\max\{Q_f(\mathbf{y}'(t_1,r_1))||r_1|\le s(2^{16}-1)\text{ and }2t_1\in S_{r_1}\}+s,$$

we may see that for each $0\le t\le s-1$ and $0\le r\le m-3$, the representative

$$f_m(\mathbf{y}'(t_1,r_1)+r_1\beta^{(k)})+\sum_{i=17}^{n}a_iP_m(x_i(t,r))$$

modulo $s(m-2)$ is smaller than or equal to $C_{\mathbf{a}}(m-2)$. Consequently, we may conclude that $F_m(\mathbf{x})$ represents every positive integer greater than $C_{\mathbf{a}}(m-2)$. $\square$

In the **Case VII**. $\mathcal{A}'(\mathbf{7})$, we treat the dropouts in $\mathcal{A}'(7)$ in Table 3.1. We assume that $(f_m+f_m^*)(\mathbf{x})=\sum_{i=1}^{16}a_iP_m(x_i)$, $f_m(\mathbf{x})=a_{i_1}P_m(x_{i_1})+\cdots+a_{i_{10}}P_m(x_{i_{10}})$ and $\beta^{(k)}\in\mathbb{Z}^{10}$ with $B_f(\alpha,\beta^{(k)})=1$ for all $1\le k\le N$. In the **Case VII**. $\mathcal{A}'(\mathbf{7})$, we show that for each $\mathbf{a}\in\mathcal{A}(7)'$, there are $8\le i_8<i_9<i_{10}\le 16$ and $\beta^{(k)}\in\mathbb{Z}^{10}(1\le k\le N)$ satisfying (3.4.4) and (3.4.5) in Lemma 3.4.2.

Case VII. $\mathcal{A}'(\mathbf{7})$

Consider $\mathbf{a} = (1, 2, 4, 7a'_4, \cdots, 7a'_{16}) \in \mathcal{A}'(7)$.

From the definition of $\mathcal{A}$, we have that

$$a'_4 = 1, \quad a'_5 \in \{1, 2\}, \quad a'_6 \in \{1, 2, 3, 4\}, \quad a'_7 \in \{1, 2, \cdots, 8\}.$$

Therefore by using the Proposition 3.3.1, we may obtain that for any odd prime $p \neq 7 =: p_0$, $L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p$ are (even) universal for $i_8 = 8, i_9 = 9, i_{10} = 10$ and

$$\beta^{(1)} := \mathbf{e}_1, \quad \beta^{(2)} := -\mathbf{e}_1 + \mathbf{e}_2, \quad \beta^{(3)} := -3\mathbf{e}_1 + \mathbf{e}_3$$

with $B_f(\alpha, \beta^{(k)}) = 1$ for all $1 \leq k \leq 3$.

And for $p = 2$, one may show that $L_{f,\beta^{(k)}} \otimes \mathbb{Z}_2$ are even universal for all $k = 1, 2, 3$ through a similar processing with **Case I**. $\mathcal{A}(\mathbf{13})$ or **Case III**. $\mathcal{A}(\mathbf{7})$.

So, the above arguments induce that (3.4.4) holds for $p_0 = 7$.

Now for $p_0 = 7$, consider $S_{f+f^*,r_1} \otimes \mathbb{Z}_7$ for $|r_1| < s(2^{16} - 1)$. When $r_1 \equiv 0 \pmod 7$, we may show that $S_{f+f^*,r_1} \otimes \mathbb{Z}_7 \subseteq \mathbb{Z}_7 \backslash (\mathbb{Z}_7^\times)^2 \subsetneq \mathbb{Z}_7$. For if

$$x_1(t_1, r_1) + 2x_2(t_1, r_1) + 4x_3(t_1, r_1) \equiv \sum_{i=1}^{16} a_i P_m(x_i) = r_1 \equiv 0 \pmod 7,$$

then we may have that $x_1(t_1, r_1) \equiv -2x_2(t_1, r_1) - 4x_3(t_1, r_1) \pmod 7$. For $t_1 \in S_{f+f^*,r_1}$, since

$$x_1(t_1, r_1)^2 + 2x_2(t_1, r_1)^2 + 4x_3(t_1, r_1)^2 \equiv \sum_{i=1}^{16} a_i (x_i(t_1, r_1)^2 - x_i(t_1, r_1)) = 2t_1 \pmod 7$$

holds, we may see that

$$\begin{aligned}
2t_1 &\equiv x_1(t_1, r_1)^2 + 2x_2(t_1, r_1)^2 + 4x_3(t_1, r_1)^2 \\
&\equiv (-2x_2(t_1, r_1) - 4x_3(t_1, r_1))^2 + 2x_2(t_1, r_1)^2 + 4x_3(t_1, r_1)^2 \\
&\equiv -x(t_1, r_1)^2 + 2x_2(t_1, r_1)x_3(t_1, r_1) - x_3(t_1, r_1)^2 \\
&= -(x_2(t_1, r_1) - x_3(t_1, r_1))^2 \pmod 7,
\end{aligned}$$

which yields $t_1 \notin (\mathbb{Z}_7^\times)^2$ because -2 is a quadratic non-residue modulo 7. On the other hand, we may use the Proposition 3.3.3 to see that

$$\begin{aligned} L_{f,\beta^{(k)}} \otimes \mathbb{Z}_7 = (\mathbb{Z}_7\alpha + \mathbb{Z}_7\beta^{(k)})^{\perp} &\cong (\langle 1\rangle \perp \langle -1\rangle)^{\perp} \\ &\cong \langle -1\rangle \perp \langle 7a_4'\rangle \perp \cdots \perp \left\langle 7a_{i_{10}}'\right\rangle \end{aligned}$$

for all $k = 1, 2, 3$. From the fact that -1 is a quadratic non-residue modulo 7 and $1 \le a_4', a_5', a_6' < 7$, we may obtain that the represented integers by $(L_{f,\beta^{(k)}} \otimes \mathbb{Z}_7, Q_f)$ are

$$Q_f(L_{f,\beta^{(k)}} \otimes \mathbb{Z}_7) = \mathbb{Z}_7 \backslash (\mathbb{Z}_7^{\times})^2 \tag{3.4.6}$$

for all $1 \le k \le 3$. By using the (5.2.7), we may obtain the followings

$$\bigcup_{k=1}^{3} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_7\} = \mathbb{Z}_7 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_7 \tag{3.4.7}$$

for $r_1 \in \mathbb{Z}_7^{\times}$ and,

$$\bigcup_{k=1}^{3} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_7\} = \mathbb{Z}_7 \backslash (\mathbb{Z}_7^{\times})^2 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_7, \tag{3.4.8}$$

for $r_1 \in 7\mathbb{Z}_7$ which imply that (3.4.5) in Lemma 3.4.2 holds for $p_0 = 7$.

Consequently, we may conclude that for $p_0 = 7$, $\beta^{(1)} := \mathbf{e}_1, \beta^{(2)} := -\mathbf{e}_1 + \mathbf{e}_2, \beta^{(3)} := -3\mathbf{e}_1 + \mathbf{e}_3$ and $(i_8, i_9, i_{10}) = (8, 9, 10)$, both of (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied.

Case VIII. $\mathcal{A}'(\mathbf{5})$

First consider $\mathbf{a} = (1, 1, 3, 5a_4', \cdots, 5a_{16}') \in \mathcal{A}'(5)$.

From the definition of $\mathcal{A}$, we have that

$$a_4' = 1, \quad a_5' \in \{1, 2\}, \quad a_6' \in \{1, 2, 3, 4\}, \quad a_7' \in \{1, 2, \cdots, 8\}.$$

We first cosider the case when $a_5' = 1$. By using the Proposition 3.3.1, we may obtain that for any odd prime $p \ne 5 =: p_0$, $L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p$ are (even) universal for $(i_8, i_9, i_{10}) = (8, 9, 10)$ and

$$\beta^{(1)} = -4\mathbf{e}_2 + \mathbf{e}_4,\ \beta^{(2)} = -5\mathbf{e}_1 - 7\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5$$

with $B_f(\alpha, \beta^{(k)}) = 1$ for all $1 \le k \le 2$.

And for $p = 2$, one may show that $L_{f,\beta^{(k)}} \otimes \mathbb{Z}_2$ are even universal for all $k = 1, 2$ through a similar processing with **Case I**. $\mathcal{A}(\mathbf{13})$ or **Case III**. $\mathcal{A}(\mathbf{7})$.

So, the above arguments induce that (3.4.4) holds for $p_0 = 5$.

Now for $p_0 = 5$, consider $S_{f+f^*,r_1} \otimes \mathbb{Z}_5$ for $|r_1| < s(2^{16} - 1)$. When $r_1 \equiv 0 \pmod 5$, we may show that $S_{f+f^*,r_1} \otimes \mathbb{Z}_5 \subseteq \mathbb{Z}_5 \backslash 2(\mathbb{Z}_5^\times)^2 \subsetneq \mathbb{Z}_5$ through a similar processing with the **Case VII**. $\mathcal{A}'(\mathbf{7})$. On the other hand, we may use the Proposition 3.3.3 to see that

$$\begin{aligned} L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5 = (\mathbb{Z}_5\alpha + \mathbb{Z}_5\beta^{(k)})^\perp &\cong (\langle 1 \rangle \perp \langle -1 \rangle)^\perp \\ &\cong \langle 3 \rangle \perp \langle 5a_4' \rangle \perp \cdots \perp \left\langle 5a_{i_{10}}' \right\rangle \end{aligned}$$

for all $k = 1, 2$. From the fact that 3 is a quadratic non-residue modulo 7 and $a_4' = a_5' = 1$, we may obtain that the represented integers by $(L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5, Q_f)$ are

$$Q_f(L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5) = \mathbb{Z}_5 \backslash (\mathbb{Z}_5^\times)^2 \tag{3.4.9}$$

for all $1 \le k \le 2$. From (3.4.9), we may obtain the followings

$$\bigcup_{k=1}^{2} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5\} = \mathbb{Z}_5 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_5 \tag{3.4.10}$$

for $r_1 \in \mathbb{Z}_5^\times$ and,

$$\bigcup_{k=1}^{2} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5\} = \mathbb{Z}_5 \backslash (\mathbb{Z}_5^\times)^2 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_5, \tag{3.4.11}$$

for $r_1 \in 5\mathbb{Z}_5$ which imply that (3.4.5) in Lemma 3.4.2 holds for $p_0 = 5$.

Consequently, we may conclude that for $p_0 = 5$, $\beta^{(1)} = -4\mathbf{e}_2 + \mathbf{e}_4$, $\beta^{(2)} = -5\mathbf{e}_1 - 7\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 \in \mathbb{Z}^{10}$, and $(i_8, i_9, i_{10}) = (8, 9, 10)$ both of (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied.

When $a'_5 = 2$, through a similar processing with the above argument when $a'_5 = 1$, one may show that for $p_0 = 5$, $\beta^{(1)} = -4\mathbf{e}_2 + \mathbf{e}_4$, $\beta^{(2)} = -5\mathbf{e}_1 - 7\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_5 \in \mathbb{Z}^{10}$ and $i_8 = 8,\ i_9 = 9,\ i_{10} = 10$, both of (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied by showing that for each $k = 1, 2$

$$\begin{cases} L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ is (even) universal} & \text{for every prime } p \neq 5 \\ L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5 \cong \langle 3\rangle \perp \langle 5a'_4\rangle \perp \cdots \left\langle 5a'_{i_{10}}\right\rangle & , \end{cases} \tag{3.4.12}$$

yielding

$$\bigcup_{k=1}^{2}\{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5\} = \mathbb{Z}_5 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_5 \tag{3.4.13}$$

for $r_1 \in \mathbb{Z}_5^{\times}$ and,

$$\bigcup_{k=1}^{2}\{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5\} = \mathbb{Z}_5 \backslash (\mathbb{Z}_5^{\times})^2 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_5 \tag{3.4.14}$$

for $r_1 \in 5\mathbb{Z}_5$.

Secondly, consider $\mathbf{a} = (1, 2, 2, 5a'_4, \cdots, 5a'_{16}) \in \mathcal{A}'(5)$.

Through a similar processing with the above argument, one may show that for $p_0 = 5$, $\beta^{(1)} = -3\mathbf{e}_3 + \mathbf{e}_5$, $\beta^{(2)} = -\mathbf{e}_1 - \mathbf{e}_2 + 6\mathbf{e}_3 - \mathbf{e}_6 \in \mathbb{Z}^{10}$ and $i_8 = 8,\ i_9 = 9,\ i_{10} = 10$, both of (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied by showing that for each $k = 1, 2$

$$\begin{cases} L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ is (even) universal} & \text{for every prime } p \neq 5 \\ L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5 \cong \langle 1\rangle \perp \langle 5a'_4\rangle \perp \cdots \left\langle 5a'_{i_{10}}\right\rangle, & \end{cases} \tag{3.4.15}$$

yielding

$$\bigcup_{k=1}^{2}\{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5\} = \mathbb{Z}_5 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_5 \tag{3.4.16}$$

for $r_1 \in \mathbb{Z}_5^\times$ and,

$$\bigcup_{k=1}^{2}\{Q_f(x)|x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5\} = \mathbb{Z}_5 \backslash 2(\mathbb{Z}_5^\times)^2 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_5 \qquad (3.4.17)$$

for $r_1 \in 5\mathbb{Z}_5$.

Finally, consider $\mathbf{a} = (1,1,3,6,10,15a'_6,15a'_7,3a'_8\cdots,3a'_{16}) \in \mathcal{A}'(5)$.

Through a similar processing with the above argument, one may show that for $p_0 = 3$, $\beta^{(1)} = -3\mathbf{e}_3 + \mathbf{e}_5$, $\beta^{(2)} = -\mathbf{e}_1 - \mathbf{e}_2 + 6\mathbf{e}_3 - \mathbf{e}_6$ and $i_8 = 8$, $i_9 = 9$, $i_{10} = 10$, both of (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied by showing that for each $k = 1, 2$

$$\begin{cases} L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ is (even) universal} & \text{for every prime } p \neq 3 \\ L_{f,\beta^{(k)}} \otimes \mathbb{Z}_5 \cong \langle 2\rangle \perp \langle 3\rangle \perp \langle 3\rangle \perp \cdots \left\langle 3a'_{i_{10}}\right\rangle, \end{cases} \qquad (3.4.18)$$

yielding

$$\bigcup_{k=1}^{2}\{Q_f(x)|x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} = \mathbb{Z}_3 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_3 \qquad (3.4.19)$$

for $r_1 \in \mathbb{Z}_3^\times$ and

$$\bigcup_{k=1}^{2}\{Q_f(x)|x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} = \mathbb{Z}_3 \backslash (\mathbb{Z}_3^\times)^2 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_3 \qquad (3.4.20)$$

for $r_1 \in 3\mathbb{Z}_3$.

Case IX. $\mathcal{A}'(\mathbf{3})$

First consider $\mathbf{a} = (1,1,3a'_3,\cdots,3a'_{16}) \in \mathcal{A}'(3)$.

One may easily show that

$$\begin{cases} S_{f+f^*,r_1} \subseteq \mathbb{Z}_3 \setminus 2(\mathbb{Z}_3^\times)^2 & \text{when } r_1 \equiv 0 \pmod 3 \\ S_{f+f^*,r_1} \subseteq \mathbb{Z}_3 \setminus (\mathbb{Z}_3^\times)^2 & \text{when } r_1 \equiv 1 \pmod 3 \\ S_{f+f^*,r_1} \subseteq \mathbb{Z}_3 \setminus 2(\mathbb{Z}_3^\times)^2 & \text{when } r_1 \equiv 2 \pmod 3 \end{cases} \qquad (3.4.21)$$

from

$$2t_1 \equiv x_1(t_1,r_1)^2 + x_2(t_1,r_1)^2 - r_1 \pmod 3$$

and

$$\begin{cases} (x_1(t_1,r_1),x_2(t_1,r_1)) \equiv (1,2),(2,1) \text{ or } (0,0) & \text{when } r_1 \equiv 0 \pmod 3 \\ (x_1(t_1,r_1),x_2(t_1,r_1)) \equiv (1,0),(0,1) \text{ or } (2,2) & \text{when } r_1 \equiv 1 \pmod 3 \\ (x_1(t_1,r_1),x_2(t_1,r_1)) \equiv (2,0),(0,2) \text{ or } (1,1) & \text{when } r_1 \equiv 2 \pmod 3 \end{cases}$$

where $t_1 \in S_{f+f^*,r_1}$ and $r_1 = \sum_{i=1}^{16} a_i x_i(t_1,r_1) \equiv x_1(t_1,r_1)+x_2(t_1,r_1) \pmod 3$.

On the other hand, one may show that

$$\begin{cases} L_{f,\beta} \otimes \mathbb{Z}_p \text{ is even universal} & \text{for every } p \neq 3 \\ L_{f,\beta} \otimes \mathbb{Z}_3 \text{ represents } \mathbb{Z}_3 \setminus (\mathbb{Z}_3^{\times})^2 & \end{cases}$$

for $(i_8,i_9,i_{10}) = (8,9,10)$ and the following β

$$\begin{cases} \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + 4\mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,3,3) \\ \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + 2\mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,3,6) \\ \beta = -4\mathbf{e}_1 - 4\mathbf{e}_2 - 4\mathbf{e}_3 - 4\mathbf{e}_4 - 4\mathbf{e}_5 + 5\mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,3,9) \\ \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + \mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,3,12) \\ \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 + 4\mathbf{e}_4 - \mathbf{e}_5 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,6,a_6) \\ \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + 2\mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,9,9) \\ \beta = -4\mathbf{e}_1 - 4\mathbf{e}_2 - 4\mathbf{e}_3 - 4\mathbf{e}_4 + 9\mathbf{e}_5 - 4\mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,9,12) \\ \beta = 2\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 + 2\mathbf{e}_4 - 5\mathbf{e}_5 + 2\mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,9,15) \\ \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + \mathbf{e}_6 & \text{when } (a_3,a_4,a_5,a_6) = (3,3,9,18) \\ \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 + 2\mathbf{e}_5 & \text{when } (a_3,a_4,a_5,a_6) = (3,6,6,a_6) \\ \beta = -4\mathbf{e}_1 - 4\mathbf{e}_2 - 4\mathbf{e}_3 - 4\mathbf{e}_4 + 5\mathbf{e}_5 & \text{when } (a_3,a_4,a_5,a_6) = (3,6,9,a_6) \\ \beta = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 + \mathbf{e}_5 & \text{when } (a_3,a_4,a_5,a_6) = (3,6,12,a_6). \end{cases} \tag{3.4.22}$$

Consequently, we may conclude that for $\mathbf{a} = (1,1,3a'_3,\cdots,3a'_{16}) \in \mathcal{A}'(3)$, for the given $\beta \in \mathbb{Z}^{10}$ in (3.4.22) and $(i_8,i_9,i_{10}) = (8,9,10)$, the local conditions (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied.

Secondly, consider $\mathbf{a} = (1, 2, 3a'_3, \cdots, 3a'_{16}) \in \mathcal{A}'(3)$.

One may show that

$$S_{f+f^*,r_1} \subseteq 3\mathbb{Z}_3 \text{ when } r_1 \equiv 0 \pmod 3 \tag{3.4.23}$$

similarly with the above from

$$2t_1 \equiv x_1(t_1, r_1)^2 + 2x_2(t_1, r_1)^2 - r_1 \pmod 3$$

where $t_1 \in S_{f+f^*,r_1}$ and $r_1 = \sum_{i=1}^{16} a_i x_i(t_1, r_1) \equiv x_1(t_1, r_1) + 2x_2(t_1, r_1) \pmod 3$.

On the other hand, one may show that

$$\begin{cases} L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ are even universal} & \text{for every } p \neq 3 \\ L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3 \text{ represents } 3\mathbb{Z}_3 \end{cases}$$

for $i_8 = 8,\ i_9 = 9,\ i_{10} = 10$ and the following $\beta^{(k)}$'s

$$\begin{cases} \beta^{(1)} = \mathbf{e}_1, \beta^{(2)} = -\mathbf{e}_2 + \mathbf{e}_3 + 2\mathbf{e}_4, \beta^{(3)} = -\mathbf{e}_2 + \mathbf{e}_3 & \text{when } (a_4, a_5, a_6) = (3, 6, 6) \\ \beta^{(1)} = \mathbf{e}_1, \beta^{(2)} = -3(\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_6) + 38\mathbf{e}_2, \beta^{(3)} = -\mathbf{e}_2 + \mathbf{e}_3 & \text{when } (a_4, a_5, a_6) = (6, 12, 21) \\ \beta^{(1)} = \mathbf{e}_1, \beta^{(2)} = -\mathbf{e}_2 + \mathbf{e}_3, \beta^{(3)} = -\mathbf{e}_2 + \mathbf{e}_3 & \text{otherwise} \end{cases} \tag{3.4.24}$$

yielding

$$\bigcup_{k=1}^{3} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} = \mathbb{Z}_3 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_3 \tag{3.4.25}$$

for $r_1 \in \mathbb{Z}_3^\times$ and

$$\bigcup_{k=1}^{3} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} = 3\mathbb{Z}_3 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_3 \tag{3.4.26}$$

for $r_1 \in 3\mathbb{Z}_3$.

Consequently, we may conclude that for $\mathbf{a} = (1, 2, 3a'_3, \cdots, 3a'_{16}) \in \mathcal{A}'(3)$, for the given $\beta^{(k)} \in \mathbb{Z}^{10}$ in (3.4.24) and $(i_8, i_9, i_{10}) = (8, 9, 10)$, the local conditions (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied.

Thirdly, we consider $(1, 1, 3a'_3, \cdots, 3a'_k, 3n+1, 3a'_{k+2}, \cdots, 3a'_{16}) \in \mathcal{A}'(3)$ for some $3 \le k+1 \le 16$.

Note that

$$(a_3, a_4, a_5) \in \{(1, 3, a_5), (3, 3, a_5), (3, 4, 6), (3, 4, 9), (3, 6, a_5)\}.$$

One may show that in the case when $(a_3, a_4, a_5) \neq (3, 4, 9)$,

$$S_{f+f^*, r_1} \subseteq \mathbb{Z}_3 \setminus 2(\mathbb{Z}_3^\times)^2 \text{ when } r_1 \equiv 0 \pmod 3. \tag{3.4.27}$$

So through a similar processing with the above, one may yield that for $p_0 = 3$, $(i_8, i_9, i_{10}) = (8, 9, 10)$ and the following $\beta^{(k)} \in \mathbb{Z}^{10}$

$$\begin{cases} \beta^{(1)} = \mathbf{e}_1, \beta^{(2)} = 6\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 & \text{when } (a_3, a_4) = (1, 3) \\ \beta^{(1)} = \mathbf{e}_1, \beta^{(2)} = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 + 2\mathbf{e}_4 & \text{when } (a_3, a_4) = (3, 3) \\ \beta^{(1)} = \mathbf{e}_1, \beta^{(2)} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 & \text{when } (a_3, a_4, a_5) = (3, 4, 6) \\ \beta^{(1)} = \mathbf{e}_1, \beta^{(2)} = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_4 & \text{when } (a_3, a_4, a_5) = (3, 6, a_5), \end{cases} \tag{3.4.28}$$

the local conditions (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied by showing that

$$L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ are even universal for every } p \neq 3, \tag{3.4.29}$$

$$\bigcup_{k=1}^{2} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} = \mathbb{Z}_3 \supseteq 2S_{f+f^*, r_1} \otimes \mathbb{Z}_3 \tag{3.4.30}$$

for $r_1 \in \mathbb{Z}_3^\times$ and

$$\bigcup_{k=1}^{2} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} = \mathbb{Z}_3 \backslash (\mathbb{Z}_3^\times)^2 \supseteq 2S_{f+f^*, r_1} \otimes \mathbb{Z}_3. \tag{3.4.31}$$

for $r_1 \in 3\mathbb{Z}_3$.

On the other hand, in the case when $(a_3, a_4, a_5) = (3, 4, 9)$, one may show that

$$\begin{cases} S_{f+f^*,r_1} \subseteq 2(\mathbb{Z}_3^\times)^2 \cup 3(\mathbb{Z}_3^\times)^2 \cup 9\mathbb{Z}_3 & \text{when } r_1 \equiv 0 \pmod 9 \\ S_{f+f^*,r_1} \subseteq 2(\mathbb{Z}_3^\times)^2 \cup 6(\mathbb{Z}_3^\times)^2 \cup 9\mathbb{Z}_3 & \text{when } r_1 \equiv 3 \pmod 9 \\ S_{f+f^*,r_1} \subseteq 2(\mathbb{Z}_3^\times)^2 \cup 3(\mathbb{Z}_3^\times) & \text{when } r_1 \equiv 6 \pmod 9. \end{cases} \tag{3.4.32}$$

In this case, one may yield that for $p_0 = 3$, $(i_8, i_9, i_{10}) = (8, 9, 10)$ and $\beta^{(1)} = \mathbf{e}_1,\ \beta^{(2)} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4,\ \beta^{(3)} = 2\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 - \mathbf{e}_5$ the local conditions (3.4.4) and (3.4.5) in Lemma 3.4.2 are satisfied by showing that

$$L_{f,\beta^{(k)}} \otimes \mathbb{Z}_p \text{ are even universal for every } p \neq 3 \tag{3.4.33}$$

$$\bigcup_{k=1}^{3} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} = \mathbb{Z}_3 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_3 \tag{3.4.34}$$

for $r_1 \in \mathbb{Z}_3^\times$ and

$$\bigcup_{k=1}^{3} \{Q_f(x) | x \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_3\} \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_3 \tag{3.4.35}$$

for $r_1 \in 3\mathbb{Z}_3$ by using the (3.4.32).

Finally, we treat the case $(1, 2, 3, 6, 8, 8a'_6, 24, 8a'_8, \cdots, 8a'_{16}) \in \mathcal{A}'(3)$ in the next section with the dropouts in $\mathcal{A}'(2)$.

3.5 $\mathcal{A}'(2)$: Exceptional candidates in $\mathcal{A}(2)$

We finally treat the dropouts in $\mathcal{A}'(2)$ to complete the Theorem 5.3.8. Throughout this section, we write $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$.

Corollary 3.5.1. *Suppose p be an odd prime.*

(1) If there are at least 6 units of $\mathbb{Z}_p$(by admitting recursion) in $\{a_{i_1}, \cdots, a_{i_{10}}\}$, then $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal for any $\beta \in \mathbb{Z}^{10}$ such that $B_f(\alpha, \beta) = 2^t$ where $t \in \mathbb{N}_0$.

(2) *If there are at least 5 units and at least 2 prime elements of $\mathbb{Z}_p$(by admitting recursion) in $\{a_{i_1}, \cdots, a_{i_{10}}\}$, then $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal for any $\beta \in \mathbb{Z}^{10}$ such that $B_f(\alpha, \beta) = 2^t$ where $t \in \mathbb{N}_0$.*

Proof. This directly follows from the Proposition 3.2.6 and Proposition 3.3.1 because

$$\mathbb{Z}_p\alpha + \mathbb{Z}_p\beta = \mathbb{Z}_p\alpha + \mathbb{Z}_p\left(\frac{1}{2^t}\beta\right)$$

and $B_f(\alpha, \frac{1}{2^t}\beta) = 1$. □

Remark 3.5.2. *Note that*

$$f_m(\mathbf{x} + r_1'\beta) = \frac{m-2}{2}\{Q_f(\mathbf{x}) + r_1'^2 Q_f(\beta) - 2^t r_1'\} + 2^t r_1'$$

where $\mathbf{x} \in L_{f,\beta}$ with $B_f(\alpha, \beta) = 2^t$.

Case X. $\mathcal{A}'(\mathbf{2}) - \mathbf{1}$
Consider $\mathbf{a} = (1, 2, 2, 5, 8a_5', \cdots, 8a_{16}') \in \mathcal{A}'(2)$.
From the definition of $\mathcal{A}$, we have that

$$a_5' = 1, \quad a_6' \in \{1, 2\}, \quad a_7' \in \{1, 2, 3, 4\}.$$

Therefore from the Corollary 3.5.1, we may have that for any odd prime p, $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal for any $8 \le i_8 < i_9 < i_{10} \le 16$ and $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha, \beta) = 2^t$ where $t \in \mathbb{N}_0$.

Let $\beta := \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 - \mathbf{e}_5 \in \mathbb{Z}^{10}$. Then $B_f(\alpha, \beta) = 2$ because $i_k = k$ for all $1 \le k \le 7$. By using the Proposition 3.3.5(with $j_1 = 1, j_2 = 2, j_3 = 3, j_4 = 4$), we may obtain that $L_{f,\beta} \otimes \mathbb{Z}_2$ is even universal for any $8 \le i_8 < i_9 < i_{10} \le 16$.

So with the above argument about the odd primes, we may obtain that $L_{f,\beta}$ is locally even universal. Therefore for each $0 \le t_1 \le s-1$ and an even $0 \le r_1 =: 2r_1' \le 2^{16} - 1$, $L_{f,\beta}$ may represent an integer which is equivalent with $2t_1 - r_1'^2 Q_f(\beta) + 2r_1'$ modulo $2s$, i.e., there is an $\mathbf{x}(t_1, r_1) \in L_{f,\beta}$ for which

$$Q_f(\mathbf{x}(t_1, r_1)) \equiv 2t_1 - r_1'^2 Q_f(\beta) + 2r_1' \pmod{2s}.$$

And then we have

$$f_m(\mathbf{x}(t_1, r_1) + r_1'\beta) \equiv t_1(m-2) + r_1 \pmod{s(m-2)}.$$

We now assume that r_1 is odd. For $\beta^{(1)} := \mathbf{e}_1, \beta^{(2)} := -\mathbf{e}_1 + \mathbf{e}_2 \in \mathbb{Z}^{10}$, by using the Proposition 3.3.5 and the Proposition 3.3.6, one may show that $L_{f,\beta^{(k)}} \otimes \mathbb{Z}_2$ represents every 2-adic even integer except 2 and 10 up to unit-square for each $k = 1, 2$, i.e., $L_{f,\beta^{(k)}}$ locally represent every even integer which is not equivalent to 2 modulo 8 for each $k = 1, 2$. So we may have that

$$\bigcup_{k=1}^{2} \{Q_f(\mathbf{x}) | \mathbf{x} \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_2\} = 2\mathbb{Z}_2$$

for each $r_1 \in \mathbb{Z}_2^\times$. Therefore for each $0 \le t_1 \le s-1$ and $0 \le r_1 \le 2^{16}-1$ with an odd r_1, there is a $k \in \{1, 2\}$ for which $L_{f,\beta^{(k)}}$ represents an integer which is equivalent with $2t_1 - r_1^2 Q_f(\beta^{(k)}) + r_1$ modulo $2s$, i.e., there is an $\mathbf{x}(t_1, r_1) \in L_{f,\beta^{(k)}}$ for which

$$Q_f(\mathbf{x}(t_1, r_1)) \equiv 2t_1 - r_1^2 Q_f(\beta^{(k)}) + r_1 \pmod{2s}.$$

And then we have

$$f_m(\mathbf{x}(t_1, r_1) + r_1\beta^{(k)}) \equiv t_1(m-2) + r_1 \pmod{s(m-2)}.$$

Consequently, we may conclude that for $\mathbf{a} = (1, 2, 2, 5, 8a_5', \cdots, 8a_{16}') \in \mathcal{A}'(2)$, a well choosen subform $f_m(\mathbf{x}) = a_{i_k} P_m(x_{i_k}) + \cdots + a_{i_{10}} P_m(x_{i_{10}})$ of $\sum_{i=1}^{16} a_i P_m(x_i)$ represents an integer which is equivalent to $t_1(m-2) + r_1$ modulo $s(m-2)$ for each $0 \le t_1 \le s-1$ and $0 \le r_1 \le 2^{16}-1$, yielding for $m \ge 2^{16}+3$, if an m-gonal form $F(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ having its first 16 coefficients as $(1, 2, 2, 5, 8a_5', \cdots, 8a_{16}') \in \mathcal{A}'(2)$ represents every positive integer up to $m-4$, then $F(\mathbf{x})$ represents every positive integer greater than $C_{\mathbf{a}}(m-2)$ for any $m \ge 2^{16}+3$ where $C_{\mathbf{a}}$ is a constant which is dependent only on $\mathbf{a} = (a_1, \cdots, a_{16})$.

Remark 3.5.3. *Through a similar processing with the* **Case X**. $\boldsymbol{\mathcal{A}'(2) - 1}$, *one may show that for each*

$$\begin{aligned}\mathbf{a} \in \{&(1,2,2,5,10,16a'_5,\cdots,16a'_{16}),(1,2,4,4,7,16a'_5,\cdots,16a'_{16}),\\ &(1,2,4,4,9,16a'_5,\cdots,16a'_{16}),(1,2,4,4,11,16a'_5,\cdots,16a'_{16}),\\ &(1,2,4,7,12,16a'_5,\cdots,16a'_{16})\}\subset \mathcal{A}'(2),\end{aligned}$$

there is a constant $C_{\mathbf{a}}$ which is dependent only on $\mathbf{a}$ for which for $m \geq 2^{16}+3$, if an m-gonal form $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ with $\mathbf{a} = (a_1,\cdots,a_{16})$ represents every positive integer up to $m-4$, then $F_m(\mathbf{x})$ represetns every positive integer greater than or equal to $C_{\mathbf{a}}(m-2)$.

On the other hand, for

$$\begin{aligned}\mathbf{a} \in \{&(1,2,3,6,8a'_5,\cdots,8a'_{16}),(1,2,4,5,12,16a'_5,\cdots,16a'_{16}),\\ &(1,2,2,3,8a'_5,\cdots,8a'_{16}),(1,2,4,4,5,16a'_6,\cdots,16a'_{16})\}\subset \mathcal{A}'(2),\end{aligned}$$

there occur an issue which is similar one occured in the case $\mathbf{a} \in \mathcal{A}'(p)$ in Section 3.4. We treat the above $\mathbf{a}$ in the next **Case XI**. $\mathcal{A}'(\mathbf{2}) - \mathbf{2}$.

Case XI. $\mathcal{A}'(\mathbf{2}) - \mathbf{2}$

Consider $\mathbf{a} = (1,2,2,3,8a'_5,\cdots,8a'_{16}) \in \mathcal{A}'(2)$.

From the definition of A, we have that

$$a'_5 = 1, \quad a'_6 \in \{1,2\}, \quad a'_7 \in \{1,2,3,4\}.$$

Therefore from the Corollary 3.5.1, we may have that for any odd prime p, $L_{f,\beta} \otimes \mathbb{Z}_p$ is (even) universal for any $8 \leq i_8 < i_9 < i_{10} \leq 16$ and $\beta \in \mathbb{Z}^{10}$ with $B_f(\alpha,\beta) = 2^t$ for $t \in \mathbb{N}_0$.

For $\beta^{(1)} := \mathbf{e}_1$ and $\beta^{(2)} := -\mathbf{e}_1 + \mathbf{e}_2$, one may see that each of $L_{f,\beta^{(1)}} \otimes \mathbb{Z}_2$ and $L_{f,\beta^{(2)}} \otimes \mathbb{Z}_2$ represents every integer in $4\mathbb{Z}_2$ for any $8 \leq i_8 < i_9 < i_{10} \leq 16$ by using the Proposition 3.3.5 and Proposition 3.3.6. From this, we may obtain that for $r_1 \in \mathbb{Z}_2^{\times}$,

$$\bigcup_{k=1}^{2}\{Q_f(\mathbf{x}) + r_1^2 \cdot Q_f(\beta^{(k)})^2 - r_1 | \mathbf{x} \in L_{f,\beta^{(k)}} \otimes \mathbb{Z}_2\} = 2\mathbb{Z}_2 \supseteq 2S_{f+f^*,r_1} \otimes \mathbb{Z}_2.$$

Thus for each $0 \le t_1 \le s-1$ and odd $0 \le r_1 \le 2^{16}-1$, there may be $\mathbf{y}(t_1, r_1) \in \mathbb{Z}^{10}$ such that

$$f_m(\mathbf{y}(t_1, r_1)) \equiv t_1(m-2) + r_1 \pmod{s(m-2)}.$$

For $\beta^{(3)} := \mathbf{e}_2$, one may see that $L_{f,\beta^{(3)}} \otimes \mathbb{Z}_2$ represents every integer in $4\mathbb{Z}_2$ for any $8 \le i_8 < i_9 < i_{10} \le 16$ by using the Proposition 3.3.5. From this we may obtain that for $r_1 =: 2r_1' \in 2\mathbb{Z}_2^{\times}$ and $L_{f,\beta^{(k)}} \otimes \mathbb{Z}_2 =: (L_{f,\beta^{(k)}})_2$

$$\{Q_f((L_{f,\beta^{(1)}})_2) + (2r_1)^2 \cdot Q_f(\beta^{(1)})^2\} \cup \{Q_f((L_{f,\beta^{(3)}})_2) + (r_1')^2 \cdot Q_f(\beta^{(3)})^2\} = 2\mathbb{Z}_2.$$

Thus for each $0 \le t_1 \le s-1$ and $0 \le r_1 \le 2^{16}-1$ with $r_1 \equiv 2 \pmod 4$, there may be $\mathbf{y}(t_1, r_1) \in \mathbb{Z}^{10}$ such that

$$f_m(\mathbf{y}(t_1, r_1)) \equiv t_1(m-2) + r_1 \pmod{s(m-2)}.$$

On the other hand, we may induce that when $r_1 \equiv 0 \pmod 4$, for $t \in S_{f+f^*,r_1}$,

$$2t + r_1 = x_1^2 + 2x_2^2 + 2x_3^2 + 3x_4^2 + 8a_5'x_5^2 + \cdots + 8a_{16}'x_{16}^2 \not\equiv 2 \pmod 8 \quad (3.5.1)$$

through similar arguments in **Case VII.**$\boldsymbol{\mathcal{A}'(7)}$ from

$$r_1 = x_1 + 2x_2 + 2x_3 + 3x_4 + 8a_5'x_5 + \cdots + 8a_{16}'x_{16} \equiv 0 \pmod 4.$$

And we may show that $L_{f,\beta^{(1)}} \otimes \mathbb{Z}_2$ represents every integer in $2\mathbb{Z}_2$ except 2 and 10 up to unit square. In other words, $L_{f,\beta^{(1)}} \otimes \mathbb{Z}_2$ represents every even integer which is not congruent with 2 modulo 8. Thus with (3.5.1) we may obtain that for $r_1 \in 4\mathbb{Z}_2$,

$$\{Q_f(\mathbf{x}) + r_1^2 \cdot Q_f(\beta^{(1)})^2 - r_1 | \mathbf{x} \in L_{f,\beta^{(1)}} \otimes \mathbb{Z}_2\} \supseteq 2S_{f.f^*,r_1} \otimes \mathbb{Z}_2$$

holds. Thus for each $\sum_{i=1}^{16}(\mathbf{x}(t,r)) =: t_1(m-2)+r_1$ where $[-\frac{m}{2}] \le r_1 \le [\frac{m-2}{2}]$ with $r_1 \equiv 0 \pmod 4$, there may be a vector $\mathbf{y}(t_1, r_1) \in \mathbb{Z}^{10}$ such that

$$f_m(\mathbf{y}(t_1, r_1)) \equiv t_1(m-2) + r_1 \pmod{s(m-2)}.$$

Consequently, we may conclude that for $\mathbf{a} = (1, 2, 2, 3, 8a'_5, \cdots, 8a'_{16}) \in \mathcal{A}'(2)$, a well choosen subform $f_m(\mathbf{x}) = a_{i_1}P_m(x_{i_1}) + \cdots + a_{i_{10}}P_m(x_{i_{10}})$ of $\sum_{i=1}^{16} a_iP_m(x_i)$ represents an integer which is equivalent to $\sum_{i=1}^{16} a_iP_m(x_i(t, r))$ $=: t_1(m-2) + r_1$ modulo $s(m-2)$ for each $0 \le t(m-2) + r < s(m-2)$, yielding for $m \ge 2^{17}s$, if an m-gonal form $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_iP_m(x_i)$ with $\mathbf{a} := (a_1, \cdots, a_{16}) = (1, 2, 2, 3, 8a'_5, \cdots, 8a'_{16}) \in \mathcal{A}'(2)$ represents every positive integer up to $s(m-2)$, then $F_m(\mathbf{x})$ represetns every positive integer greater than $C_{\mathbf{a}}(m-2)$ for any $m \ge 2^{17}s$ where $C_{\mathbf{a}}$ is a constant which is dependent only on $\mathbf{a} = (a_1, \cdots, a_{16})$.

Remark 3.5.4. *For each the remaining*

$$\begin{aligned}\mathbf{a} \in \{&(1, 2, 3, 6, 8a'_5, \cdots, 8a'_{16}), (1, 2, 4, 5, 12, 16a'_5, \cdots, 16a'_{16}),\\ &(1, 2, 4, 4, 5, 16a'_6, \cdots, 16a'_{16})\} \subset \mathcal{A}'(2),\end{aligned}$$

one may show that there is a constant $C_{\mathbf{a}}$ which is dependent only on $\mathbf{a}$ *for which for $m \ge 2^{17}s$, if an m-gonal form $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_iP_m(x_i)$ with* $\mathbf{a} = (a_1, \cdots, a_{16})$ *represents every positive integer up to $s(m-2)$, then $F_m(\mathbf{x})$ represetns every positive integer greater than or equal to $C_{\mathbf{a}}(m-2)$ for any $m \ge 2^{17}s$ through a similar processing with the* **Case XI**. $\mathcal{A}'(\mathbf{2}) - \mathbf{2}$. *I leave the remainings to reader.*

Through the calculations, we may finally get the following main theorem in this chapter.

Theorem 3.5.5. *For $m \ge 3$, there exists an absolute constant C such that*

$$\gamma_m \le C(m-2). \tag{3.5.2}$$

Proof. Put $C := \max(\{\frac{\gamma_m}{m-2} | 3 \le m \le 2^{17}s\} \cup \{C_{\mathbf{a}} | \mathbf{a} \in \mathcal{A}\})$. $\square$

Remark 3.5.6. *We may observe that the absolute constant C constructed as a maximum through this chapter in Theorem 3.5.5 is so huge even though there could exist much smaller optimal C satisfying (3.5.2) in the Theorem 3.5.5. But in the Chapter 4 and Chapter 5, we adopt the huge C in applying the Theroem 3.5.5.*

Chapter 4

The minimal rank for universal m-gonal form

In this chapter, we show that when m is sufficiently large, the minimal rank for universal m-gonal form is $\lceil \log_2(m-3) \rceil$ for most m and $\lceil \log_2(m-3) \rceil + 1$ for the rest of m.

4.1 Prerequisite for universality

In Theorem 2.4.3 (3), we showed the requirement (2.4.2) for an m-gonal form to represent every positive integer up to only $m-4$. And in order to follow the requirement (2.4.2), it is required at least $\lceil \log_2(m-3) \rceil$ variables. Namely, the rank of a universal m-gonal form would be greater than or equal to $\lceil \log_2(m-3) \rceil$. On the other hand, we may see that an escalator $a_1 P_m(x_1) + \cdots + a_n P_m(x_n)$ following the requirement (2.4.2) of rank $\lceil \log_2(m-3) \rceil$ is of the form

$$P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1} P_m(x_n) \tag{4.1.1}$$

where $n = \lceil \log_2(m-3) \rceil$ which is the most fastly escalated m-gonal form to represent every positive integer up to $m-4$.

In this chapter, we show that for most m, the m-gonal form of (4.1.1) is universal, yielding the m-gonal form is one of a universal m-gonal forms of the minimal rank. And for the rest m, we show that first every m-gonal form following the (2.4.2) of rank $\lceil \log_2(m-3) \rceil$ is not universal, then the m-gonal form

$$P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n) \tag{4.1.2}$$

where $n = \lceil \log_2(m-3) \rceil + 1$ is universal, yielding the m-gonal form of (4.1.2) is a universal m-gonal form of the minimal rank.

4.2 The most fastly escalated universal m-gonal form

Lemma 4.2.1. *For $m > 2\left(\left(2C + \frac{1}{4}\right)^{\frac{1}{4}} + \sqrt{2}\right)^2$ with $2 \le 2^{\lceil \log_2(m-3) \rceil} - m$, the m-gonal form*

$$F_m(\mathbf{x}) = P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n) \tag{4.2.1}$$

where $n = \lceil \log_2(m-3) \rceil$ is universal.

Proof. In virtue of the Theorem 3.5.5, it may be enough to show that $F_m(\mathbf{x})$ represents every positive integer up to only $C(m-2)$. Throughout this proof, we write the integers in $[1, C(m-2)]$ as

$$A(m-2) + B$$

where $0 \le A \le C$ and $0 \le B \le m-3$.

Note that

$$\begin{aligned} F_m(\mathbf{x}) =& (m-2)\{P_3(x_1-1) + 2P_3(x_2-1) + 4P_3(x_3-1) + 8P_3(x_4-1)\} \\ &+ (x_1 + 2x_2 + 4x_3 + 8x_4) + 16P_m(x_5) + \cdots + 2^{n-1}P_m(x_n). \end{aligned}$$

For a non-negative integer A, let $x(A)$ be the largest positive integer satisfying

$$P_3(x(A)-1) \le A,$$

i.e., the integer in the interval $(\sqrt{2A + \frac{1}{4}} - \frac{1}{2}, \sqrt{2A + \frac{1}{4}} + \frac{1}{2}]$. On the other hand, there is unique

$$y_1(A,B) \in \{x(A), x(A)-1, x(A)-2, x(A)-3\}$$

satisfying

$$\begin{cases} P_3(y_1(A,B)-1) \equiv A \pmod 2 \\ y_1(A,B) \equiv B \pmod 2. \end{cases}$$

In virtue of the well known fact that the ternary triangular form

$$P_3(x) + 2P_3(y) + 4P_3(z)$$

is universal, the even integer $A - P_3(y_1(A,B)-1)$ may be written as

$$2P_3(y_1(A,B)-1) + 4P_3(y_2(A,B)-1) + 8P_3(y_3(A,B)-1)$$

for some $y_i(A,B) \in \mathbb{Z}$, i.e.,

$$A = P_3(y_1(A,B)-1) + 2P_3(y_1(A,B)-1) + 4P_3(y_2(A,B)-1) + 8P_3(y_3(A,B)-1).$$

Beside that by using the fact $P_3(x) = P_3(-x-1)$, if it is necessary, by changing $y_i(A,B)-1$ to $-y_i(A,B)$, we may additionally obtain the followings

$$\begin{cases} A = P_3(y_1(A,B)-1) + \cdots + 8P_3(y_4(A,B)-1) \\ B \equiv y_1(A,B) + 2y_2(A,B) + 4y_3(A,B) + 8y_4(A,B) \pmod{16} \end{cases} \tag{4.2.2}$$

hold for some $y_i(A,B) \in \mathbb{Z}$. Since the integer $y_1(A,B)$ in (4.2.2) is in $[\sqrt{2A+\frac{1}{4}} - \frac{7}{2}, \sqrt{2A+\frac{1}{4}} + \frac{1}{2}]$ we may get that

$$\begin{aligned} 0 &\le A - P_3(y_1(A,B)-1) \\ &= 2P_3(y_2(A,B)-1) + 4P_3(y_3(A,B)-1) + 8P_3(y_4(A,B)-1) \\ &< 4\sqrt{2A+\frac{1}{4}} - 8. \end{aligned}$$

By arranging the above inequality, we may obtain

$$\left(y_2(A,B)-\frac{1}{2}\right)^2+2\left(y_3(A,B)-\frac{1}{2}\right)^2+4\left(y_4(A,B)-\frac{1}{2}\right)^2<4\sqrt{2A+\frac{1}{4}}-\frac{25}{4}. \tag{4.2.3}$$

From the (4.2.3), through a basic calculation, we may get that

$$|2y_2(A,B)+4y_3(A,B)+8y_4(A,B)|<14\sqrt{\frac{4\sqrt{2A+\frac{1}{4}}-\frac{25}{4}}{73}}.$$

So with $y_1(A,B)\in[\sqrt{2A+\frac{1}{4}}-\frac{7}{2},\sqrt{2A+\frac{1}{4}}+\frac{1}{2}]$, we may get that such the above $y_i(A,B)$ where $160\le A\le C$ satisfy

$$\begin{aligned}0&<\sqrt{2A+\frac{1}{4}}-14\sqrt{\frac{4\sqrt{2A+\frac{1}{4}}-\frac{25}{4}}{73}}-\frac{7}{2}\\&<y_1(A,B)+2y_2(A,B)+4y_3(A,B)+8y_4(A,B)\\&<\sqrt{2A+\frac{1}{4}}+14\sqrt{\frac{4\sqrt{2A+\frac{1}{4}}-\frac{25}{4}}{73}}+\frac{1}{2}<\frac{m-2}{2}.\end{aligned}$$

Through similar processings with the above we may obtain the followings

$$\begin{cases}A=P_3(z_1(A,B)-1)+\cdots+8P_3(z_4(A,B)-1)\\B\equiv z_1(A,B)+2z_2(A,B)+4z_3(A,B)+8z_4(A,B)\pmod{16}\end{cases}$$

hold for some $z_1(A,B)\in\{-x(A)+1,-x(A)+2,-x(A)+3,-x(A)+4\}$ and $z_i(A,B)\in\mathbb{Z}$ for $i=2,3,4$. And in this case, we may get that the above

$z_i(A,B)$ with $160 \le A \le C$ satisfy the following

$$\begin{aligned}-\frac{m-2}{2} &< -\sqrt{2A-\frac{1}{4}} - 14\sqrt{\frac{4\sqrt{2A+\frac{1}{4}}-\frac{25}{4}}{73}} + \frac{1}{2} \\ &\le z_1(A,B) + 2z_2(A,B) + 4z_3(A,B) + 8z_4(A,B) \\ &< -\sqrt{2A+\frac{1}{4}} + 14\sqrt{\frac{4\sqrt{2A+\frac{1}{4}}-\frac{25}{4}}{73}} + \frac{9}{2} < 0.\end{aligned}$$

And then one may easily see that for each integer $A(m-2)+B \in [160(m-2), C(m-2)]$,

$$\text{either} \ \ (x_i(A,B)) = (y_i(A,B)) \ \ \text{or} \ \ (x_i(A,B)) = (z_i(A,B))$$

satisfies

$$0 \le A(m-2) + B - \{P_m(x_1(A,B)) + \cdots + 8P_m(x_4(A,B))\} \le m-11$$

with $A(m-2) + B - \{P_m(x_1(A,B)) + \cdots + 8P_m(x_4(A,B))\} \equiv 0 \pmod{16}$. On the other hand, remaining $16P_m(x_5) + \cdots + 2^{n-1}P_m(x_n)$ may represent all the multiples of 16 up to $m-11(\le 2^n - 16)$ by taking $P_m(x_i) \in \{0,1\}$ for all $5 \le i \le n$ which yields that $A(m-2)+B$ may be represented by $F_m(\mathbf{x})$ as follows

$$P_m(x_1(A,B)) + \cdots + 8P_m(x_4(A,B)) + 16P_m(x_5) + \cdots + 2^{n-1}P_m(x_n)$$

for some $(x_5, \cdots, x_n) \in \{0,1\}^{n-4}$. Until now, we showed that

$$F_m(\mathbf{x}) = P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n)$$

represents every positive integer in $[160(m-2), C(m-2)]$.

From now on, in the remaining of this proof, we show that $F_m(\mathbf{x})$ represents every positive integer in $[1, 160(m-2)]$. Through direct calcuations(the author used python), we may obtain that for each $(A, r_B) \in \mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ with

$0 \le A \le 160$, there are integer solutions $(x_1, x_2, x_3) \in \mathbb{Z}^3$ for both of

$$\begin{cases} P_3(x_1-1) + 2P_3(x_2-1) + 4P_3(x_3-1) = A \\ x_1 + 2x_2 + 4x_3 \equiv r_B \pmod 8 \\ 0 \le x_1 + 2x_2 + 4x_3 < 100 \ll \frac{m-2}{2} \end{cases} \tag{4.2.4}$$

and

$$\begin{cases} P_3(x_1-1) + 2P_3(x_2-1) + 4P_3(x_3-1) = A \\ x_1 + 2x_2 + 4x_3 \equiv r_B \pmod 8 \\ -\frac{m-2}{2} \ll -100 < x_1 + 2x_2 + 4x_3 \le 0 \end{cases} \tag{4.2.5}$$

respectively except the pairs (A, r_B) in $S^+ \cup S^- (\subset \mathbb{Z} \times \mathbb{Z}/8\mathbb{Z})$ where

$$\begin{array}{rllllll} S^+ := & \{(0,0), & (0,1), & (0,2), & (0,3), & (0,4), & (0,5), \\ & (0,6), & (0,7), & (1,0), & (1,1), & (1,2), & (1,3), \\ & (1,4), & (1,5), & (1,6), & (2,0), & (2,1), & (2,2), \\ & (2,3), & (2,4), & (2,5), & (3,1), & (3,2), & (3,3), \\ & (3,4), & (3,7), & (4,0), & (4,1), & (4,2), & (4,3), \\ & (5,1), & (5,2), & (5,7), & (6,0), & (6,1), & (6,6), \\ & (7,0), & (7,5), & (8,0), & (8,1), & (8,2), & (8,4), \\ & (8,5), & (8,6), & (9,1), & (9,3), & (9,4), & (10,2), \\ & (10,5), & (10,7), & (11,0), & (11,3), & (11,4), & (11,5), \\ & (11,6), & (12,4), & (13,0), & (14,1), & (14,3), & (15,1), \\ & (15,2), & (16,1), & (16,2), & (16.3), & (17,3), & (17,6), \\ & (18,0), & (18,1), & (18,2), & (19,0), & (19,2), & (20,7), \\ & (21,0), & (21,1), & (22,0), & (22,1), & (23,1), & (23,5), \\ & (25,2), & (26,7), & (28,4), & (18,6), & (29,2), & (31,3), \\ & (32,2), & (32,3), & (35,2), & (36,1), & (37,1), & (37,2), \\ & (37,7), & (38,0), & (43,1), & (43,4), & (44,0), & (44,1), \\ & (50,2), & (51,6), & (53,0), & (53,3), & (53,5), & (54,7), \\ & (58,2), & (64,2), & (64,3), & (65,1), & (65,4), & (72,2), \\ & (74,2), & (75,0), & (81,1), & (85,5), & (92,4), & (106,2), \\ & (106,3), & (110,2), & (116,3), & (123,1), & (128,2), & (128,3)\} \end{array} \tag{4.2.6}$$

and

$$\begin{aligned} S^- := \{ &(1,7), && (2,6), && (2,7), && (4,4), && (4,5), && (4,6), \\ &(4,7), && (7,7), && (8,3), && (8,7), && (9,6), && (10,0), \\ &(11,2), && (11,7), && (15,5), && (18,6), && (18,7), && (20,0), \\ &(22,7), && (23,6), && (26,0), && (28,1), && (31,4), && (36,6), \\ &(37,0), && (37,5), && (44,6), && (44,7), && (53,2), && (53,4), \\ &(53,7), && (54,0), && (106,4), && (110,5), && (116,4), && (128,4), \\ &(128,5)\}. \end{aligned} \tag{4.2.7}$$

Each pair $(A, r_B) \in S^+$ has integer solutions for only (4.2.4) and not for (4.2.5) and each pair $(A, r_B) \in S^-$ has integer solutions for only (4.2.5) and not for (4.2.4). And then, we may easily see that for each

$$1 \le A(m-2) + B \le 160(m-2)$$

with $(A, r_B) \notin S^+ \cup S^-$ where r_B is the residue of B modulo 8 one of the above integer solutions $(x_1, x_2, x_3) \in \mathbb{Z}^3$ for (4.2.4) or (4.2.5) satisfies the followings

$$\begin{cases} A(m-2) + B \equiv P_m(x_1) + 2P_m(x_2) + 4P_m(x_3) \pmod 8 \\ 0 \le A(m-2) + B - \{P_m(x_1) + 2P_m(x_2) + 4P_m(x_3)\} < m-2. \end{cases} \tag{4.2.8}$$

Denote an integer solution for (4.2.8) by $(x_1(A,B), x_2(A,B), x_3(A,B)) \in \mathbb{Z}^3$. Since the remaining

$$8P_m(x_4) + \cdots + 2^{n-1}P_m(x_n)$$

represents

$$A(m-2) + B - \{P_m(x_1(A,B)) + 2P_m(x_2(A,B)) + 4P_m(x_3(A,B))\}$$

(which is a multiple of 8 in $[0, m-3]$ from (4.2.8)) by taking $P_m(x_i) \in \{0,1\}$ for all $4 \le i \le n$, we may obtain that every integer $A(m-2) + B$ in

$[1, 160(m-2)]$ with $(A, r_B) \notin S^+ \cup S^-$ is represented by

$$F_m(\mathbf{x}) = P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n)$$

To complete the proof, we finally consider the integers

$$A(m-2) + B \in [1, 160(m-2)]$$

with $(A, r_B) \in S^+ \cup S^-$ where r_B is the residue of B modulo 8. Among the pairs (A, r_B) in $S^+ \cup S^-$, consider $(116, 3) \in S^+$. From

$$P_m(1) + 2P_m(-7) + 4P_m(6) = 116(m-2) + 11, \tag{4.2.9}$$

we may yield that every positive integer $A(m-2)+B \in [1, 160(m-2)]$ with $(A, r_B) = (116, 3)$ may be represented by $F_m(\mathbf{x})$ other than $116(m-2)+3$. On the other hand, $116(m-2)+3$ may be equivalent with one $115(m-2)+B'$ of the

$$\begin{cases} 115(m-2) + 32 = P_m(-2) + 2P_m(1) + 4P_m(8) \\ 115(m-2) + 41 = P_m(3) + 2P_m(9) + 4P_m(5) \\ 115(m-2) + 42 = P_m(2) + 2P_m(6) + 4P_m(7) \\ 115(m-2) + 35 = P_m(3) + 2P_m(0) + 4P_m(8) \\ 115(m-2) + 36 = P_m(-2) + 2P_m(9) + 4P_m(5) \\ 115(m-2) + 37 = P_m(11) + 2P_m(1) + 4P_m(6) \\ 115(m-2) + 38 = P_m(2) + 2P_m(10) + 4P_m(4) \\ 115(m-2) + 39 = P_m(-1) + 2P_m(6) + 4P_m(7) \end{cases}$$

modulo 8 because $32, 41, 42, 35, 36, 37, 38,$ and 39 form a complete set of residues modulo 8. By agian using the fact that the remaining $8P_m(x_4) + \cdots + 2^{n-1}P_m(x_n)$ represents all the multiples of 8 up to $m-3$, we may obtaint that $116(m-2)+3$ is also represented by $F_m(\mathbf{x})$ because

$$(116(m-2)+3) - (115(m-2)+B')$$

would be a multiple of 8 in $[0, m-3]$. Similarly with the above case $(A, r_B) = (116, 3) \in S^+ \cup S^-$, through surveys based on the result of integer solutions for (4.2.4) and (4.2.5) with the fact that the remaining m-gonal subform $8P_m(x_4)+\cdots+2^{n-1}P_m(x_n)$ of $F_m(\mathbf{x})$ represents all the multiples of 8 up to $m-3(\le 2^n-8)$, we may get that every integer $A(m-2)+B$ in $[1,160(m-2)]$ with $(A,r_B)\in S^+\cup S^-$ other than

$$\begin{matrix} 37(m-2)+7, & 37(m-2)+15, & 37(m-2)+m-18, \\ 37(m-2)+m-16, & 37(m-2)+m-11, & 37(m-2)+m-10 \end{matrix} \tag{4.2.10}$$

may be represented by $F_m(\mathbf{x})$ by taking $(x_4,\cdots,x_n)\in\{0,1\}^{n-3}$. In the meanwhile, we may directly check that the representability of integers in (4.2.10) by $F_m(\mathbf{x})$ as follows

$$\begin{cases} 37(m-2)+7=P_m(-1)+2P_m(4)+4P_m(0)+8P_m(-2)+16P_m(1) \\ 37(m-2)+15=P_m(-1)+2P_m(0)+4P_m(-2)+8P_m(3) \\ 37(m-2)+m-18=P_m(4)+2P_m(0)+4P_m(-3)+8P_m(-1) \\ 37(m-2)+m-16=P_m(4)+2P_m(1)+4P_m(-3)+8P_m(-1) \\ 37(m-2)+m-11=P_m(-3)+2P_m(-3)+4P_m(-2)+8P_m(-1)+16P_m(1) \\ 37(m-2)+m-10=P_m(-4)+2P_m(0)+4P_m(-1)+8P_m(-2)+16P_m(1). \end{cases}$$

Consequently, we may conclude that

$$F_m(\mathbf{x})=P_m(x_1)+2P_m(x_2)+\cdots+2^{n-1}P_m(x_n)$$

represents every positive integer up to $C(m-2)$, yielding the $F_m(\mathbf{x})$ is universal by the Theorem 3.5.5.

□

Remark 4.2.2. *In the Lemma 4.2.1, we proved that for sufficiently large m with $2\le 2^{\lceil \log_2(m-3)\rceil}-m$,*

$$r_m=\lceil \log_2(m-3)\rceil$$

by showing that

$$P_m(x_1)+2P_m(x_2)+4P_m(x_3)+\cdots+2^{n-1}P_m(x_n)$$

where $n = \lceil \log_2(m-3) \rceil$ is universal. Now we consider the r_m for $-3 \le 2^{\lceil \log_2(m-3) \rceil} - m \le 4$.

(1) When $-3 = 2^{\lceil \log_2(m-3) \rceil} - m$, there is only one m-gonal form which represents every positive integer up to $m-4$ with the rank $n = \lceil \log_2(m-3) \rceil$ that is

$$P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n)$$

which does not represent $m-2$, i.e., there is no m-gonal form which represents every positive integer up to $m-2$ of the rank less than or equal to $\lceil \log_2(m-3) \rceil$. Clearly we may have that there is no universal m-gonal form of the rank less than or equal to $\lceil \log_2(m-3) \rceil$ when $-3 = 2^{\lceil \log_2(m-3) \rceil} - m$ which implies that

$$\lceil \log_2(m-3) \rceil + 1 \le r_m.$$

(2) When $-2 = 2^{\lceil \log_2(m-3) \rceil} - m$, all of the m-gonal forms which represent every positive integer up to $m-4$ with the rank $n = \lceil \log_2(m-3) \rceil$ are

$$\begin{cases} P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n) \text{ and} \\ P_m(x_1) + 2P_m(x_2) + \cdots + (2^{n-1}-1)P_m(x_n) \end{cases}$$

which do not represent $m-2$. So in this case too, we may have that

$$\lceil \log_2(m-3) \rceil + 1 \le r_m.$$

(3) When $-1 = 2^{\lceil \log_2(m-3) \rceil} - m$, all of the m-gonal forms which represent every positive integer up to $m-4$ with the rank $n = \lceil \log_2(m-3) \rceil$ are

$$\begin{cases} P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + (2^{n-1}-1)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + (2^{n-1}-2)P_m(x_n) \end{cases}$$

which do not represent $2m-4$, $m-2$, and $m-2$, respectively which yields that

$$\lceil \log_2(m-3) \rceil + 1 \le r_m.$$

(4) When $0 = 2^{\lceil \log_2(m-3)\rceil} - m$, the all of m-gonal forms which represent every positive integer up to $m-4$ with the rank $n = \lceil \log_2(m-3)\rceil$ are

$$\begin{cases} P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + 2^{n-1}P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + (2^{n-1}-1)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + (2^{n-1}-2)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + (2^{n-1}-3)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + (2^{n-2}-1)P_m(x_{n-1}) + (2^{n-1}-2)P_m(x_n) \end{cases}$$

which do not represent $2m-3$, $2m-4$, $m-2$, $m-2$, and $m-2$, respectively which yields that

$$\lceil \log_2(m-3)\rceil + 1 \le r_m.$$

(5) When $1 = 2^{\lceil \log_2(m-3)\rceil} - m$, all of the m-gonal forms which represent every positive integer up to $m-4$ with the rank $n = \lceil \log_2(m-3)\rceil$ are

$$\begin{cases} P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + 2^{n-1}P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + (2^{n-1}-1)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + (2^{n-1}-2)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + (2^{n-1}-3)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-2}P_m(x_{n-1}) + (2^{n-1}-4)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + (2^{n-2}-1)P_m(x_{n-1}) + (2^{n-1}-2)P_m(x_n) \\ P_m(x_1) + 2P_m(x_2) + \cdots + (2^{n-2}-1)P_m(x_{n-1}) + (2^{n-1}-3)P_m(x_n) \end{cases}$$

which do not represent $5(m-2)-1$, $2m-3$, $2m-4$, $m-2$, $m-2$, $m-2$, and $m-2$, respectively which yields that

$$\lceil \log_2(m-3)\rceil + 1 \le r_m.$$

(6) When $2 \le 2^{\lceil \log_2(m-3)\rceil} - m \le 4$, one may see that

$$P_m(x_1) + 2P_m(x_2) + \cdots + 2^{n-1}P_m(x_n)$$

where $n = \lceil \log_2(m-3) \rceil$ is universal by showing that the m-gonal form represents every positive integer up to $C(m-2)$ through similar processings with the Lemma 4.2.1. But it would require more delicate care to examine the representability of small integers in $[1, 160(m-2)]$ than the Lemma 4.2.1 because of the tighter condition of integers represented by $8P_m(x_4) + \cdots + 2^{n-1}P_m(x_n)$. The proof is omitted here.

Lemma 4.2.3. *For $m > 2\left(\left(2C + \frac{1}{4}\right)^{\frac{1}{4}} + \sqrt{2}\right)^2$, the m-gonal form*

$$F_m(\mathbf{x}) = P_m(x_1) + 2P_m(x_2) + \cdots + 2^n P_m(x_{n+1}) \tag{4.2.11}$$

where $n = \lceil \log_2(m-3) \rceil$ is universal.

Proof. One may prove this lemma through almost same arguments with the proof of Lemma 4.2.1. Actually, under the assumption of this lemma, the fact that the subform

$$8P_m(\mathbf{x}_4) + \cdots + 2^n P_m(\mathbf{x}_{n+1})$$

of $F_m(\mathbf{x})$ represents all the multiples of 8 up to $2m - 14$ instead of $m - 3$ make to show this lemma a lot easier than to show the Lemma 4.2.1. $\square$

Remark 4.2.4. *The Lemma 4.2.3 says that $F_m(\mathbf{x})$ is a universal m-gonal form of the rank $\lceil \log_2(m-3) \rceil + 1$. On the other hand, we observed that there is no universal m-gonal form of the rank less than or equal to $\lceil \log_2(m-3) \rceil$ when $-3 \le 2^{\lceil \log_2(m-3) \rceil} - m \le 1$ in the Remark 4.2.2. Therefore we may obtain that the $F_m(\mathbf{x})$ in the Lemma 4.2.3 is a universal m-gonal form of the minimal rank when $-3 \le 2^{\lceil \log_2(m-3) \rceil} - m \le 1$ which yields that $r_m = \lceil \log_2(m-3) \rceil + 1$. With the Lemma 4.2.1, Remark 4.2.2, and Lemma 4.2.3, we may conclude the following theorem.*

Theorem 4.2.5. *For $m > 2\left(\left(2C + \frac{1}{4}\right)^{\frac{1}{4}} + \sqrt{2}\right)^2$,*

$$r_m = \begin{cases} \lceil \log_2(m-3) \rceil + 1 & \text{when } -3 \le 2^{\lceil \log_2(m-3) \rceil} - m \le 1 \\ \lceil \log_2(m-3) \rceil & \text{when } \quad 2 \le 2^{\lceil \log_2(m-3) \rceil} - m. \end{cases} \tag{4.2.12}$$

Chapter 5

The maximal rank of proper universal m-gonal forms

5.1 Integers which an escalator represents

Proposition 5.1.1. *An escalator $F_m(\mathbf{x}) = \sum_{i=1}^{k} a_i P_m(x_i)$ represents every positive integer up to $\sum_{i=1}^{k} a_i$.*

Proof. The proof proceeds by induction on k.

When $k = 1$, it is clear because $a_1 = 1$.

And now assume that the Proposition is true for all escalators of rank $k-1$, that is, any escalator $\sum_{i=1}^{k-1} a_i P_m(x_i)$ of rank $k-1$ represents every positive integer up to $\sum_{i=1}^{k-1} a_i$. For a escalator $\sum_{i=1}^{k} a_i P_m(x_i)$ of the rank k to obtain a contradiction, assume that there is an integer $\alpha \leq \sum_{i=1}^{k} a_i$ which is not represented by the escalator $\sum_{i=1}^{k} a_i P_m(x_i)$. And then since the truant of $\sum_{i=1}^{k-1} a_i P_m(x_i)$ is less than the truant of $\sum_{i=1}^{k} a_i P_m(x_i)$(which is less than or equal to α), we may get that $a_k \leq \alpha$ because a_k must be less than or equal to the truant of $\sum_{i=1}^{k-1} a_i P_m(x_i)$. Since $0 \leq \alpha - a_k \leq \sum_{i=1}^{k-1} a_i$, by the induction hypothesis, $\alpha - a_k$ may be represented by $\sum_{i=1}^{k-1} a_i P_m(x_i)$. Therefore

$$\alpha = (\alpha - a_k) + a_k$$

may be represented by $\sum_{i=1}^{k} a_i P_m(x_i)$ by taking $P_m(x_k) = 1$, which is a contradiction. This completes the proof. □

Remark 5.1.2. *We may have that a node*

$$\sum_{i=1}^{k} a_i P_m(x_i)$$

with $a_1 + \cdots + a_k \geq C(m-2)$ would be a leaf, i.e., a universal escalator by the Theorem 3.5.5 and the Proposition 5.1.1

Lemma 5.1.3. *For $m > 6C^2(C+1)$, a leaf $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ with $a_{l_m} \geq C+1$ where $l_m := \left\lfloor \frac{m-2}{C+1} \right\rfloor$ has the rank $n \leq \left(1 - \frac{1}{(C+1)^2}\right)(m-2) + \frac{C+2}{C+1}$.*

Proof. To obtain a contradiction assume that $n > (1 - \frac{1}{(C+1)^2})(m-2) + \frac{C+2}{C+1}$. Then we may get

$$\begin{aligned}\sum_{i=1}^{n-1} a_i &= \sum_{i=1}^{l_m} a_i + \sum_{i=l_m+1}^{n-1} a_i \\ &> \left(\frac{m-2}{C+1} - 1\right) + (C+1)\left(n - 1 - \frac{m-2}{C+1}\right) > C(m-2)\end{aligned}$$

which yields the parent $\sum\limits_{i=1}^{n-1} a_i P_m(x_i)$ of $F_m(\mathbf{x})$ is already universal by the Remark 5.1.2, which is a contradiction. Consequently, we may conclude that $n \leq (1 - \frac{1}{(C+1)^2})(m-2) + \frac{C+2}{C+1}$. $\square$

Remark 5.1.4. *Now we consider a escalator $\sum\limits_{i=1}^{n} a_i P_m(x_i)$ with $0 \neq a_{l_m} < C+1$ where $l_m := \left\lfloor \frac{m-2}{C+1} \right\rfloor$. One may easily see that there appear 5 consecutively same coefficients between C-th coefficient a_C and $5C$-th coefficient a_{5C}, i.e., there is $C \leq t \leq 5C-4$ for which*

$$a_t = a_{t+1} = \cdots = a_{t+4}$$

because there are $4C+1$ components between $a_C P_m(x_C)$ and $a_{5C} P_m(x_{5C})$ and $1 \leq a_C \leq a_{C+1} \leq \cdots \leq a_{5C} \leq a_{l_m} < C+1$.

5.2 Escalators having same 5 coefficients

Lemma 5.2.1. *For $A \in \mathbb{N}$, the m-gonal form*

$$\sum_{i=1}^{5} A \cdot P_m(x_i)$$

represents all the multiples of $A(m-2)$.

Proof. See Lemma 2.2 in [1] $\square$

Proposition 5.2.2. *For $m \geq 6C^2(C+1)$, let $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ be a leaf with $0 \neq a_{l_m} < C+1$ where $l_m := \left\lfloor \frac{m-2}{C+1} \right\rfloor$. For $C \leq t \leq 5C-4$ such that $A := a_t = a_{t+1} = \cdots = a_{t+4}$, we rearrange the coefficients of $F_m(\mathbf{x})$ except the 5 coefficients $a_t, \cdots, a_{t+4}$ as follows*

$$b_i := \begin{cases} a_i & \text{when } i < t \\ a_{i+5} & \text{when } i \geq t. \end{cases}$$

Then we have the followings.

(1) For $i \leq l_m$, the inequality $b_i \leq b_1 + \cdots + b_{i-1} + 1$ always holds.

(2) If there is $l_m < i \leq C \cdot l_m$ such that $b_i > b_1 + \cdots + b_{i-1} + 1$, then $n < m-4$.

Proof. (1) For $i < t$, if $C \leq b_1 + \cdots + b_{i-1}$, then since we have that $b_i = a_i \leq a_{l_m} \leq C$ from the assumption, we may get that $b_i \leq b_1 + \cdots + b_{i-1} + 1$. If $a_1 + \cdots + a_{i-1} = b_1 + \cdots + b_{i-1} < C < m-4$, then the truant of the node $a_1 P_m(x_1) + \cdots + a_{i-1} P_m(x_{i-1})$ would be $a_1 + \cdots + a_{i-1} + 1 (< m-3)$ by the fact that the smallest m-gonal number is $m-3$ except 0 and 1 and Proposition 5.1.1. So we may obtain that $b_i = a_i \leq a_1 + \cdots + a_{i-1} + 1 = b_1 + \cdots + b_{i-1} + 1$.

If there is $t \leq i \leq l_m$ such that $b_i > b_1 + \cdots + b_{i-1} + 1$. Then we may get that

$$C \leq t \leq i = (i-1) + 1 \leq b_1 + \cdots + b_{i-1} + 1 < b_i \leq a_{l_m}.$$

This is a contradiction to $a_{l_m} \leq C$. This yields the claim.

(2) To obtain a contradiction, assume that $n \geq m-4$. If there is $l_m \left(= \lfloor \frac{m-2}{C+1} \rfloor\right) < i \leq C \cdot l_m$ such that $b_i > b_1 + \cdots + b_{i-1} + 1$, we may have that

$$a_{i+5} > a_1 + \cdots + a_{i+4} + 1 - 5A \geq i + 5 - 5A > \frac{m-2}{C+1} - 5A \geq (C+1)^2.$$

And then from

$$\begin{aligned}\sum_{i=1}^{n-1} a_i = \sum_{i=1}^{C \cdot l_m + 4} a_i + \sum_{i=C \cdot l_m + 5}^{n-1} a_i \\ &\geq (C \cdot l_m + 4) + (C+1)^2(n - 1 - C \cdot l_m - 4) \\ &\geq (C \cdot l_m + 4) + (C+1)^2(m - 9 - C \cdot l_m) > C(m-2),\end{aligned}$$

we may obtain a contradiction that the parent $\sum_{i=1}^{n-1} a_i P_m(x_i)$ of $F_m(\mathbf{x})$ is already universal by the Remark 5.1.2. Consequently, we may conclude that $n < m-4$. □

Remark 5.2.3. *In Lemma 5.1.3 and Proposition 5.2.2, we showed that the rank of leaf under some conditions does not exceed $m-4$. Actually, what the conditions have in common is that its escalating isn't happening that slowly. Under such the conditions, we reach to leaf node, i.e., a universal escalator before $m-4$ escalating steps.*

Lemma 5.2.4. *For $m > 6C^2(C+1)$, let $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ be a leaf with $0 \neq a_{l_m} < C+1$ where $l_m := \lfloor \frac{m-2}{C+1} \rfloor$. Then*

$$(a_1, a_2) = (1,1) \text{ or } (a_1, a_2, a_3) \in \{(1,2,2), (1,2,3), (1,2,4)\}$$

and we may take $C \leq t \leq 5C-4$ such that

$$a_t = a_{t+1} = \cdots = a_{t+4} =: A.$$

(1) When $(a_1, a_2) = (1,1)$, if there is $C \leq t \leq 5C-4$ for which $a_t = \cdots = a_{t+4} = A > 6$, then $n < m-4$.

(2) When $(a_1, a_2, a_3) = (1,2,2)$, if there is $C \leq t \leq 5C-4$ for which

$a_t = \cdots = a_{t+4} = A > 12$, *then* $n < m-4$.

(2) *When* $(a_1, a_2, a_3) = (1, 2, 3)$, *if there is* $C \le t \le 5C - 4$ *for which* $a_t = \cdots = a_{t+4} = A > 12$, *then* $n < m - 4$.

(4) *When* $(a_1, a_2, a_3) = (1, 2, 4)$, $n < m - 4$.

Proof. Since the coefficients of $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$ follow the conditions

$$\begin{cases} a_1 = 1 \\ a_{i+1} \le a_1 + \cdots + a_i + 1 \quad \text{if } a_1 + \cdots + a_i < m - 4, \end{cases} \tag{5.2.1}$$

we may have that

$$(a_1, a_2) = (1, 1), \text{ or } (a_1, a_2, a_3) \in \{(1, 2, 2), (1, 2, 3), (1, 2, 4)\}.$$

We prove this lemma only the case $(a_1, a_2) = (1, 1)$ and omit the proof of (2),(3) and (4) in this thesis. One may prove those through similar arguments with the proof of (1).

Under same notation as in Proposition 5.2.2, in virtue of the Proposition 5.2.2, we could assume that

$$b_i \le b_1 + \cdots + b_{i-1} + 1 \tag{5.2.2}$$

for all $1 \le i \le \min\{n - 5, C \cdot l_m\}$. Under the assumption (5.2.2), we prove that $n \le C \cdot l_m + 5 \left(\approx \frac{C}{C+1} m\right) < m - 4$. To obtain a contradiction, assume that $n > C \cdot l_m + 5$. From

$$\begin{aligned} \sum_{i=1}^{C\cdot l_m} b_i = \sum_{i=1}^{5C} b_i + \sum_{i=5C+1}^{C\cdot l_m} b_i &\ge 5C + A(C \cdot l_m - 5C) \\ &\ge 5C + AC(\frac{m-2}{C+1} - 1) - 5AC \\ &> (A-1)(m-2) + 4, \end{aligned}$$

with (5.2.2), we may yield that

$$\sum_{i=1}^{C\cdot l_m} b_i P_m(x_i)$$

represents every positive integer up to $(A-1)(m-2)+4$ by taking $P_m(x_i) \in \{0,1\}$ for all i. And for integer $N \in [(A-1)(m-2)+5, A(m-2)]$, since

$$0 < N - \{(6(m-2)+4\} < (A-1)(m-2),$$

it may be written as

$$N - \{6(m-2)+4\} = \sum_{i=1}^{C\cdot l_m} b_i P_m(N(x_i)) \tag{5.2.3}$$

where $(N(x_i))_i \in \{0,1\}^{C\cdot l_m}$ from the above argument. Up to reordering, we may assume that $(N(x_1), N(x_2)) = (0,0), (1,0)$, or $(1,1)$. Note that

$$\begin{cases} P_m(4) + P_m(0) = 6(m-2)+4 \\ P_m(4) + P_m(1) = 6(m-2)+5 \\ P_m(3) + P_m(3) = 6(m-2)+6. \end{cases} \tag{5.2.4}$$

And then with (5.2.3) and (5.2.4), we may see that the integer $N \in [(A-1)(m-2)+5, A(m-2)]$ is written as follows

$$\begin{cases} N = P_m(4) + P_m(0) + \sum\limits_{i=3}^{C\cdot l_m} b_i P_m(N(x_i)) & \text{when } (N(x_1), N(x_2)) = (0,0) \\ N = P_m(4) + P_m(1) + \sum\limits_{i=3}^{C\cdot l_m} b_i P_m(N(x_i)) & \text{when } (N(x_1), N(x_2)) = (1,0) \\ N = P_m(3) + P_m(3) + \sum\limits_{i=3}^{C\cdot l_m} b_i P_m(N(x_i)) & \text{when } (N(x_1), N(x_2)) = (1,1). \end{cases}$$

As a result, we may obtain that $\sum_{i=1}^{C\cdot l_m} b_i P_m(x_i)$ represents every positive integer up to $A(m-2)$.

On the other hand, by the Lemma 5.2.1, all the multiples of $A(m-2)$

may be represented by

$$\sum_{i=t}^{t+4} a_i P_m(x_i).$$

Consequently, we may conclude that

$$\sum_{i=1}^{C\cdot l_m+5} a_i P_m(x_i) = \sum_{i=1}^{C\cdot l_m} b_i P_m(x_i) + \sum_{i=t}^{t+4} a_i P_m(x_i)$$

represents every positive integer, i.e., the parent of $F_m(\mathbf{x})$ is already universal, yielding a contradiction to $F_m(\mathbf{x})$ is a leaf. This completes the proof of (1).

By using the below equations instead of (5.2.4), one may show (2), (3), and (4) through similar arguments with the above.

(2) When $(a_1, a_2, a_3) = (1, 2, 2)$, one may use following equations

$$\begin{cases} P_m(0) + 2P_m(-2) + 2P_m(-2) = 12m - 32 \\ P_m(-3) + 2P_m(-2) + 2P_m(0) = 12m - 31 \\ P_m(-4) + 2P_m(-1) + 2P_m(0) = 12m - 30 \\ P_m(-3) + 2P_m(-2) + 2P_m(1) = 12m - 29 \\ P_m(0) + 2P_m(-3) + 2P_m(1) = 12m - 28 \\ P_m(1) + 2P_m(-3) + 2P_m(1) = 12m - 27. \end{cases} \tag{5.2.5}$$

(3) When $(a_1, a_2, a_3) = (1, 2, 3)$, one may use following equations

$$\begin{cases} P_m(-2) + 2P_m(-2) + 3P_m(-1) = 12m - 33 \\ P_m(-2) + 2P_m(0) + 3P_m(-2) = 12m - 32 \\ P_m(-3) + 2P_m(-2) + 3P_m(0) = 12m - 31 \\ P_m(0) + 2P_m(-3) + 3P_m(0) = 12m - 30 \\ P_m(1) + 2P_m(-3) + 3P_m(0) = 12m - 29 \\ P_m(3) + 2P_m(-2) + 3P_m(-1) = 12m - 28 \\ P_m(3) + 2P_m(0) + 3P_m(-2) = 12m - 27. \end{cases} \tag{5.2.6}$$

(4) When $(a_1, a_2, a_3) = (1, 2, 4)$, one may use following equations(note that when $a_3 = 4$, $A \geq 4$)

$$\begin{cases} P_m(2)+2P_m(-1)+4P_m(0)=3m-6 \\ P_m(-1)+2P_m(-1)+4P_m(1)=3m-5 \\ P_m(-2)+2P_m(0)+4P_m(1)=3m-4 \\ P_m(3)+2P_m(0)+4P_m(0)=3m-3 \\ P_m(-2)+2P_m(1)+4P_m(1)=3m-2 \\ P_m(3)+2P_m(1)+4P_m(0)=3m-1 \\ P_m(2)+2P_m(2)+4P_m(0)=3m \\ P_m(3)+2P_m(0)+4P_m(1)=3m+1. \end{cases} \tag{5.2.7}$$

□

Remark 5.2.5. *In Lemma 5.2.4, we showed that if a leaf* $F_m(\mathbf{x})=\sum\limits_{i=1}^{n}a_iP_m(x_i)$ *with* $0\neq a_{l_m}<C+1$ *has the 5 consecutively same coefficients* A *greater than* 12 *between* C*-th component and* $5C$*-th component, then its rank could not exceed* $m-4$*. Especially under the condition (5.2.2), the rank would be less than or equal to* $C\cdot l_m+5(\approx \frac{C}{C+1}m)$ *(more stirict calculations would much reduce the upper bound for rank* n*).*

On the other hand, next lemma may help to consider the upper bound for rank of leaves $F_m(\mathbf{x})=\sum\limits_{i=1}^{n}a_iP_m(x_i)$ *with* $0\neq a_{l_m}<C+1$ *for which every 5 consecutively same coefficients appearing between* C*-th component and* $5C$*-th component is less than or equal to* 12*.*

Lemma 5.2.6. *Let* $F_m(\mathbf{x})=\sum\limits_{i=1}^{n}a_iP_m(x_i)$ *be a leaf with* $0\neq a_{l_m}<C+1$ *where* $l_m:=\left\lfloor\frac{m-2}{C+1}\right\rfloor$*. Then we may take* $C\leq t\leq 5C-4$ *such that*

$$a_t=a_{t+1}=\cdots=a_{t+4}=:A.$$

For $A\geq 2$*, let* $i(A)$ *be the smallest index satisfying* $A-1\leq a_1+\cdots+a_{i(A)}$*.*

(1) If $a_{i(A)+1}\leq A-1$ *when* $A\geq 2$*, then* $n<m-4$*.*

(2) If $A+1\leq a_{(C-A-1)l_m+5}$*, then* $n<m-4$*.*

Proof. (1) Similarly with the Proposition 5.2.2, we rearrange the coefficients of $F_m(\mathbf{x})$ except 6 coefficients $a_{i(A)+1}, a_t, \cdots, a_{t+4}$ as follows

$$c_i := \begin{cases} a_i & \text{when } i \le i(A) \\ a_{i+1} & \text{when } i(A)+1 \le i < t-1 \\ a_{i+6} & \text{when } t-1 \le i. \end{cases}$$

Through similar arguments with the proof of Proposition 5.2.2, one may induce that if $c_i > c_1 + \cdots + c_{i-1} + 1$ for some $i \le C \cdot l_m$, then $n < m-4$. From now on, we prove the lemma under the assumption

$$c_i \le c_1 + \cdots + c_{i-1} + 1 \text{ for any } i \le C \cdot l_m. \tag{5.2.8}$$

To obtain a contradiction assume that $m-4 \le n$. Through similar arguments with the proof of Lemma 5.2.4, one may obtain that $\sum_{i=1}^{C\cdot l_m} c_i P_m(x_i)$ represents every positive integer up to $(A-1)(m-2)$ by taking $P_m(x_i) \in \{0,1\}$ for all i. And for an integer $N \in [(A-1)(m-2), A(m-2)]$, we may see that

$$0 < N - a_{i(A)+1} P_m(x_{i(A)+1}) < (A-1)(m-2) \tag{5.2.9}$$

holds for some $x_{i(A)+1} \in \{-1, 2\}$. By the above argument, since the $N - a_{i(A)+1} P_m(x_{i(A)+1})$ of (5.2.9) is represented by $\sum_{i=1}^{C\cdot l_m} c_i P_m(x_i)$, we may yield that N is represented by $\sum_{i=1}^{C\cdot l_m} c_i P_m(x_i) + a_{i(A)+1} P_m(x_{i(A)+1})$ by taking $x_{i(A)+1} \in \{-1, 2\}$. So we may obtain that

$$\sum_{i=1}^{C\cdot l_m} c_i P_m(x_i) + a_{i(A)+1} P_m(x_{i(A)+1})$$

represents every positive integer up to $A(m-2)$.

On the other hand, by Lemma 5.2.1, all the multiples of $A(m-2)$ may be represented by $\sum_{i=t}^{t+4} a_i P_m(x_i)$. Finally, we may conclude that

$$\sum_{i=1}^{C\cdot l_m + 6} a_i P_m(x_i) = a_{i(A)+1} P_m(x_{i(A)+1}) + \sum_{i=1}^{C\cdot l_m} c_i P_m(x_i) + \sum_{i=t}^{t+4} a_i P_m(x_i)$$

$\left(C \cdot l_m + 6 \approx \frac{C}{C+1}m\right)$ is already universal, yielding a contradiction to the fact that $F_m(\mathbf{x})$ is a leaf. This completes the proof of (1).

(2) In virtue of the Proposition 5.2.2, under same notation as in the Proposition 5.2.2, we could assume that

$$b_i \le b_1 + \cdots + b_{i-1} + 1 \tag{5.2.10}$$

for all $1 \le i \le \min\{n-5, C \cdot l_m\}$. Under the assumption (5.2.10), we prove that $n \le C \cdot l_m + 5 < m-4$. To obtain a contradiction, assume that $n > C \cdot l_m + 5$. With (5.2.10), from

$$\begin{aligned}
\sum_{i=1}^{C\cdot l_m} b_i &= \sum_{i=1}^{5C} b_i + \sum_{i=5C+1}^{(C-A-1)\cdot l_m - 1} b_i + \sum_{i=(C-A-1)l_m}^{C\cdot l_m} b_i \\
&\ge 5C + A((C-A-1)l_m - 1 - 5C) + (A+1)(A+1)l_m \\
&= 5C + A(C\cdot l_m - 1) + (A+1)l_m \\
&> 5C + A\left(C \cdot \frac{m-2}{C+1} - 2\right) + (A+1)\left(\frac{m-2}{C+1} - 1\right) \\
&= 5C + A(m-2) - A\left(\frac{m-2}{C+1} + 2\right) + (A+1)\left(\frac{m-2}{C+1} - 1\right) \\
&> A(m-2),
\end{aligned}$$

we may induce that $\sum_{i=1}^{C\cdot l_m} b_i P_m(x_i)$ represents every positive integer up to $A(m-2)$. So we may conclude that

$$\sum_{i=1}^{C\cdot l_m+5} a_i P_m(x_i) = \sum_{i=1}^{C\cdot l_m} b_i P_m(x_i) + \sum_{i=t}^{t+4} a_i P_m(x_i)$$

is already universal, yielding a contradiction since $\sum_{i=t}^{t+4} a_i P_m(x_i)$ represents all the multiples of $A(m-2)$ by the Lemma 5.2.1. This completes the proof. □

Remark 5.2.7. *Until now, we showed that a great part of universal escalator has rank smaller than $m-4$, more precisely, every universal escalator other*

Table 5.1: $(A; a_1, \cdots, a_{i(A)})$

A	$(a_1, \cdots, a_{i(A)})$
1	
2	(1)
3	$(1,1), (1,2)$
4	$(1,1,1), (1,1,2), (1,1,3), (1,2)$
5	$(1,1,1,1), (1,1,1,2), (1,1,1,3), (1,1,1,4), (1,2,2), (1,2,3), (1,2,4)$
6	$(1,1,1,1,1), (1,1,1,1,2), (1,1,1,1,3), (1,1,1,1,4), (1,1,1,1,5),$ $(1,1,1,2), (1,1,1,3), (1,1,1,4), (1,1,2,2), (1,1,2,3), (1,1,2,4), (1,1,2,5),$ $(1,2,2), (1,2,3), (1,2,4)$
7	$(1,2,2,2), (1,2,2,3), (1,2,2,4), (1,2,2,5), (1,2,2,6), (1,2,3)$
8	$(1,2,2,2), (1,2,2,3), (1,2,2,4), (1,2,2,5), (1,2,2,6), (1,2,3,3), (1,2,3,4),$ $(1,2,3,5), (1,2,3,6), (1,2,3,7)$
9	$(1,2,2,2,2), (1,2,2,2,3), (1,2,2,2,4), (1,2,2,2,5), (1,2,2,2,6), (1,2,2,2,7),$ $(1,2,2,2,8), (1,2,2,3), (1,2,2,4), (1,2,2,5), (1,2,2,6), (1,2,3,3), (1,2,3,4),$ $(1,2,3,5), (1,2,3,6), (1,2,3,7)$
10	$(1,2,2,2,2), (1,2,2,2,3), (1,2,2,2,4), (1,2,2,2,5), (1,2,2,2,6), (1,2,2,2,7),$ $(1,2,2,2,8), (1,2,2,3,3), (1,2,2,3,4), (1,2,2,3,5), (1,2,2,3,6), (1,2,2,3,7),$ $(1,2,2,3,8), (1,2,2,3,9), (1,2,2,4), (1,2,2,5), (1,2,2,6), (1,2,3,3), (1,2,3,4),$ $(1,2,3,5), (1,2,3,6), (1,2,3,7)$
11	$(1,2,2,2,2,2), (1,2,2,2,2,3), (1,2,2,2,2,4), (1,2,2,2,2,5), (1,2,2,2,2,6),$ $(1,2,2,2,2,7), (1,2,2,2,2,8), (1,2,2,2,2,9), (1,2,2,2,2,10),$ $(1,2,2,2,3), (1,2,2,2,4), (1,2,2,2,5), (1,2,2,2,6), (1,2,2,2,7),$ $(1,2,2,2,8), (1,2,2,3,3), (1,2,2,3,4), (1,2,2,3,5), (1,2,2,3,6), (1,2,2,3,7),$ $(1,2,2,3,8), (1,2,2,3,9), (1,2,2,4,4), (1,2,2,4,5), (1,2,2,4,6), (1,2,2,4,7),$ $(1,2,2,4,8), (1,2,2,4,9), (1,2,2,4,10), (1,2,2,5), (1,2,2,6),$ $(1,2,3,3,3), (1,2,3,3,4), (1,2,3,3,5), (1,2,3,3,6), (1,2,3,3,7),$ $(1,2,3,3,8), (1,2,3,3,9), (1,2,3,3,10), (1,2,3,4), (1,2,3,5), (1,2,3,6),$ $(1,2,3,7)$
12	$(1,2,2,2,2,2), (1,2,2,2,2,3), (1,2,2,2,2,4), (1,2,2,2,2,5), (1,2,2,2,2,6),$ $(1,2,2,2,2,7), (1,2,2,2,2,8), (1,2,2,2,2,9), (1,2,2,2,2,10)$ $(1,2,2,2,3,3), (1,2,2,2,3,4), (1,2,2,2,3,5), (1,2,2,2,3,6), (1,2,2,2,3,7),$ $(1,2,2,2,3,8), (1,2,2,2,3,9), (1,2,2,2,3,10), (1,2,2,2,3,11), (1,2,2,2,4),$ $(1,2,2,2,5), (1,2,2,2,6), (1,2,2,2,7), (1,2,2,2,8), (1,2,2,3,3), (1,2,2,3,4),$ $(1,2,2,3,5), (1,2,2,3,6), (1,2,2,3,7), (1,2,2,3,8), (1,2,2,3,9), (1,2,2,4,4),$ $(1,2,2,4,5), (1,2,2,4,6), (1,2,2,4,7), (1,2,2,4,8), (1,2,2,4,9), (1,2,2,4,10),$ $(1,2,2,5,5), (1,2,2,5,6), (1,2,2,5,7), (1,2,2,5,8), (1,2,2,5,9), (1,2,2,5,10),$ $(1,2,2,5,11), (1,2,2,6), (1,2,3,3,3), (1,2,3,3,4), (1,2,3,3,5), (1,2,3,3,6),$ $(1,2,3,3,7), (1,2,3,3,8), (1,2,3,3,9), (1,2,3,3,10), (1,2,3,4,4),$ $(1,2,3,4,5), (1,2,3,4,6), (1,2,3,4,7), (1,2,3,4,8), (1,2,3,4,9),$ $(1,2,3,4,10), (1,2,3,4,11), (1,2,3,5), (1,2,3,6), (1,2,3,7)$

than $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$ *with*

$$\begin{cases} (A; a_1, \cdots, a_{i(A)}) \text{ in Table 5.1 and} \\ a_i = A \quad \text{for all } i(A) < i \le \min\{n, (C-A-1)l_m\} \end{cases} \tag{5.2.11}$$

has the rank less than $m-4$*. So we lastly consider the rank of universal escalators* $F_m(\mathbf{x})$ *with (5.2.11). One may notice that the universal escalators* $F_m(\mathbf{x})$ *with (5.2.11) are very slowly escalated universal escalators.*

5.3 Proper universal m-gonal forms with maximal rank : Very slowly escalated universal escalators

Lemma 5.3.1. *The* m*-gonal forms*

$$a_1P_m(x_1) + \cdots + a_{i(A)}P_m(x_{i(A)}) + AP_m(x_{i(A)+1}) + \cdots + AP_m(x_n)$$

where $n = \left\lceil \left(1-\frac{1}{2A}\right)(C+1)l_m + \left(1-\frac{1}{2A}\right)(C+1)\right\rceil + i(A) + 5$ *and* $(A; a_1, \cdots, a_{i(A)})$ *are in Table 5.1 with* $A \neq 1, 3$ *are universal.*

Proof. Similarly with the Proposition 5.2.2, we rearrange the coefficients except the 5 coefficients $a_{i(A)}, a_t, \cdots, a_{t+4}$ like

$$b_i := \begin{cases} a_i & \text{when } i \le i(A) \\ a_{i+5} & \text{when } i \ge i(A)+1. \end{cases}$$

From

$$\begin{aligned}\sum_{i=1}^{n} b_i &= \sum_{i=1}^{i(A)} b_i + \sum_{i=i(A)+1}^{n-5} b_i \\ &= \sum_{i=1}^{i(A)} b_i + \sum_{i=i(A)+1}^{n-5} A \\ &\geq \sum_{i=1}^{i(A)} b_i + \left(A-\frac{1}{2}\right)(m-2),\end{aligned}$$

we may have that $\sum_{i=1}^{n-5} b_i P_m(x_i)$ represents every positive integer up to $\left\lfloor \left(A-\frac{1}{2}\right)(m-2)\right\rfloor$. On the other hand, one may directly check that for each $(A;a_1,\cdots,a_{i(A)}) = (A;b_1,\cdots,b_{i(A)})$ in Table 5.1 other than $(1;\ ),(3;1,1)$, and $(3;1,2)$,

$$a_1P_m(x_1)+\cdots+a_{i(A)}P_m(x_{i(A)}) = b_1P_m(x_1)+\cdots+b_{i(A)}P_m(x_{i(A)})$$

represents complete residues modulo A in $[m-3, \left\lfloor \left(A-\frac{1}{2}\right)(m-2)\right\rfloor]$. For example, for $(A;a_1,\cdots,a_{i(A)}) = (2;1)$, we have complete system of residues

$$P_m(-1) = m-3, \quad P_m(2) = m$$

modulo $A = 2$ and for $(A;a_1,\cdots,a_{i(A)}) = (4;1,1,1)$, we have complete system of residues

$$\begin{aligned}&P_m(-1)+P_m(0)+P_m(0) = m-3, \quad P_m(-1)+P_m(1)+P_m(0) = m-2,\\ &P_m(-1)+P_m(1)+P_m(1) = m-1, \quad P_m(2)+P_m(0)+P_m(0) = m\end{aligned}$$

modulo $A = 4$. One may without difficulty check the remaining cases too by hand similarly with the above. I omit the leangthy calcuations in this thesis. And then an integer $N \in [\left\lfloor \left(A-\frac{1}{2}\right)(m-2)\right\rfloor, A(m-2)]$ may be written as

$$N = b_1P_m(N_1)+\cdots+b_{i(A)}P_m(N_{i(A)})+AP_m(N_{i(A)+1})+\cdots+AP_m(N_{n-5})$$

where $N' = b_1 P_m(N_1) + \cdots + b_{i(A)} P_m(N_{i(A)})$ is an integer in $[m-3, (A-1)(m-2)]$ which is equivalent with N modulo A and $(N_{i(A)+1}, \cdots, N_{n-5}) \in \{0,1\}^{n-i(A)-5}$ since

$$0 \le N - N' \le (A-1)(m-2) + 1$$

is a multiple of A. So we may get that $\sum_{i=1}^{n-5} b_i P_m(x_i)$ represents every positive integer up to $A(m-2)$.

On the other hand, by the Lemma 5.2.1, since

$$AP_m(x_{i(A)+1}) + \cdots + AP_m(x_{i(A)+5})$$

represents all the multiples of $A(m-2)$, we may conclude that

$$\sum_{i=1}^{n} a_i P_m(x_i) = AP_m(x_{i(A)+1}) + \cdots + AP_m(x_{i(A)+5}) + \sum_{i=1}^{n-5} b_i P_m(x_i)$$

is universal. $\square$

Corollary 5.3.2. *Let $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ be a leaf with $0 \ne a_{l_m} < C+1$ where $l_m := \left\lfloor \frac{m-2}{C+1} \right\rfloor$. If there is $t \le 5C-4$ for which*

$$a_t = a_{t+1} = \cdots = a_{t+4} =: A.$$

with $A \ne 1, 3$, then we have $n < m-4$.

Proof. By using the Lemma 5.2.2, Lemma 5.2.6, and Lemma 5.3.1, we may yield this. $\square$

Remark 5.3.3. *More delicate care may reduce the upper bound for the rank of $F_m(\mathbf{x}) := a_1 P_m(x_1) + \cdots + a_{i(A)} P_m(x_{i(A)}) + AP_m(x_{i(A)+1}) + \cdots + AP_m(x_n)$*

$$n \approx \left(1 - \frac{1}{2A}\right)(m-2)$$

which makes

$$a_1 P_m(x_1) + \cdots + a_{i(A)} P_m(x_{i(A)}) + AP_m(x_{i(A)+1}) + \cdots + AP_m(x_n)$$

universal in Lemma 5.3.1. Actually, by Theorem 1.1 in [1], especially for $(A; a_1, \cdots, a_{i(A)})$ of the form $(A; 1, \cdots, 1)$ with $A \neq 1, 3$, we could take the below n

$$n = \begin{cases} \lfloor \frac{m}{2} \rfloor & \text{when } A = 2 \\ \lceil \frac{m-2}{4} \rceil + 2 & \text{when } A = 4 \\ \lceil \frac{m-3}{A} \rceil + (A-2) & \text{when } 5 \leq A \leq 12 \end{cases} \tag{5.3.1}$$

instead of $n = \lceil \left(1 - \frac{1}{2A}\right)(C+1)(l_m+1) \rceil + i(A) + 5 \approx \left(1 - \frac{1}{2A}\right) m$ in Lemma 5.3.1 and the n of (5.3.1) would be optimal. The author guess that for the most $(A; a_1, \cdots, a_{i(A)})$ in Table 5.1 the optimal n which makes

$$a_1 P_m(x_1) + \cdots + a_{i(A)} P_m(x_{i(A)}) + A P_m(x_{i(A)+1}) + \cdots + A P_m(x_n)$$

universal would be close to $\frac{m}{A}$ but not all.

Now we consider the rank of leaves $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$ with

$$(a_1, \cdots, a_{(C-2)l_m+5}) = (1, \cdots, 1)$$

and

$$(a_1, \cdots, a_{\min\{n,(C-4)l_m+5\}}) = (1, 1, 3. \cdots, 3), (1, 2, 3, \cdots, 3).$$

Actually among the above leaves, the leaves of the maximal rank R_m appear.

Lemma 5.3.4. *Let $F_m(\mathbf{x}) = \sum_{i=1}^{n} a_i P_m(x_i)$ be a node of the escalator tree with*

$$a_1 = a_2 = \cdots = a_{(C-2)l_m+5} = 1. \tag{5.3.2}$$

If

$$a_1 + \cdots + a_{n-1} < m - 4 \leq a_1 + a_2 + \cdots + a_n, \tag{5.3.3}$$

then the escalator would be universal, i.e., the node would become a leaf of the tree.

Proof. Note that the m-gonal form

$$f_m(\mathbf{x}) := a_6 P_m(x_6) + \cdots + a_n P_m(x_n)$$

represents every positive integer up to $m-3$ except at most 5 integers by taking $P_m(x_6) \in \{0, 1, m-3\}$ and $P_m(x_i) \in \{0, 1\}$ for all $7 \le i \le n$ and the integers not represented by $f_m(\mathbf{x})$ would be consecutive. On the specific, if there is an integer not represented by $f_m(\mathbf{x})$, then the integer always would be in $[(C-2)l_m + 1, m-4]$ and such case would occur only when

$$\begin{cases} a_1 + \cdots + a_n - 5 < m - 4 & \text{or} \\ a_n > a_1 + \cdots + a_{n-1} + 1 - 5. \end{cases} \tag{5.3.4}$$

When $a_1 + \cdots + a_n - 5 < m - 4$, the consecutive integers not represented by $f_m(\mathbf{x})$ in $[(C-2)l_m + 1, m-4]$ would be

$$a_1 + \cdots + a_n - 4, a_1 + \cdots + a_n - 3, \cdots, m - 4$$

and when $a_n > a_1 + \cdots + a_{n-1} + 1 - 5$, the consecutive integers not represented by $f_m(\mathbf{x})$ in $[(C-2)l_m + 1, m-4]$ would be

$$a_1 + \cdots + a_{n-1} - 4, a_1 + \cdots + a_{n-1} - 3, \cdots, a_n - 1.$$

In the cases that $f_m(\mathbf{x})$ represents every positive integer up to $m-3$, we may conclude that $F_m(\mathbf{x})$ is universal by using the fact that

$$a_1 P_m(x_1) + \cdots + a_5 P_m(x_5) = P_m(x_1) + \cdots + P_m(x_5)$$

represents all the multiples of $m-2$ from Lemma 5.2.1.

For the other cases, let

$$E_1 < E_2 (= E_1 + 1) < \cdots < E_s (= E_1 + (s-1))$$

where $s \le 5$ be all of the positive integers which are not represented by $f_m(\mathbf{x})$ in $[(C-2)l_m + 1, m-4]$. And then by using the fact that $a_1 P_m(x_1) + \cdots + a_5 P_m(x_5)$ represents all the multiples of $m-2$ from Lemma 5.2.1, we may yield that $F_m(\mathbf{x})$ represents every positive integer which is not congruent to $E_1, E_2, \cdots, E_s$ modulo $m-2$. On the other hand, one may observe that $E_1 - 1$ is always written as $f_m(\mathbf{x}) = a_6 P_m(x_6) + \cdots + a_n P_m(x_n)$ by taking $P_m(x_i) = 1$ for all $6 \le i \le n-1$ so in the representation of $E_1 - 1$, by changing $P_m(x_6)=1$ to $P_m(2)=m$, we may obtain

$$E_1+(m-2)=P_m(2)+P_m(0)+P_m(0)+\cdots+P_m(x_n),$$

by changing both of $P_m(x_6)=P_m(x_7)=1$ to $P_m(2)=m$, we may obtain

$$E_2+2(m-2)=P_m(2)+P_m(2)+P_m(0)+\cdots+P_m(x_n),$$

and by changing all of $P_m(x_6)=P_m(x_7)=P_m(x_8)=1$ to $P_m(2)=m$, we may obtain

$$E_3+3(m-2)=P_m(2)+P_m(2)+P_m(2)+\cdots+P_m(x_n),$$

i.e., $E_1+(m-2), E_2+2(m-2)$ and $E_3+3(m-2)$ may be represented by $f_m(\mathbf{x})$. And E_s+1 may be written as $f_m(\mathbf{x})=a_6P_m(x_6)+\cdots+a_nP_m(x_n)$ by taking $P_m(x_i)=0$ for all $7\le i\le n-1$ so in the representation of E_s+1, by changing $P_m(x_7)=0$ to $P_m(-1)=m-3$, we may obtain

$$E_s+(m-2)=P_m(x_6)+P_m(-1)+P_m(0)+P_m(0)+\cdots+P_m(x_n)$$

and by changing both of $P_m(x_7)=P_m(x_8)=0$ to $P_m(-1)=m-3$, we may obtain

$$E_{s-1}+2(m-2)=P_m(x_6)+P_m(-1)+P_m(-1)+P_m(0)+\cdots+P_m(x_n),$$

i.e., $E_s+(m-2)$ and $E_{s-1}+2(m-2)$ may be represented by $f_m(\mathbf{x})$. So by using the fact that $a_1P_m(x_1)+\cdots+a_5P_m(x_5)$ represents all the multiples of $m-2$ again, we may conclude that every positive integer except the below at most 9 positive integers

$$\begin{array}{ccccc} E_1 & E_2 & \cdots & E_{s-1} & E_s \\ & E_2+(m-2) & \cdots & E_{s-1}+(m-2) & \\ & & E_3+2(m-2) & & \end{array} \tag{5.3.5}$$

where $s\le 5$ is represented by $F_m(\mathbf{x})$.

On the other hand, the representability of the above integers in (5.3.5) by $F_m(\mathbf{x})$ may be directly confirmed. Since $a_1 + \cdots + a_n \geq m-4$, by the Proposition 5.1.1, the integers

$$E_1, E_2 \cdots, E_s$$

smaller than $m-4$ are represented by the node $F_m(\mathbf{x})$ by taking $P_m(x_i) \in \{0,1\}$ for all i and moreover, in this situation, we may additionally assume that $P_m(x_1) = P_m(x_2) = 1$. And then from a representation of $E_1, \cdots, E_{s-2}$ by $F_m(\mathbf{x})$ with $P_m(x_1) = 1$ by changing $P_m(x_1) = 1$ to $P_m(2) = m$, we may see that

$$E_2 + (m-2), \cdots, E_{s-1} + (m-2)$$

are represented by $F_m(\mathbf{x})$ and from a representation of E_1 by $F_m(\mathbf{x})$ with $P_m(x_1) = P_m(x_2) = 1$ by changing both of $P_m(x_1) = P_m(x_1) = 1$ to $P_m(2) = m$, we may see that

$$E_3 + 2(m-2)$$

is represented by $F_m(\mathbf{x})$. This completes the proof. $\square$

Remark 5.3.5. *Following the Guy's argument [6], the total sum of all coefficients of a leaf* $\sum_{i=1}^{n} a_i P_m(x_i)$ *must exceed* $m-4$*, i.e.,* $a_1 + \cdots + a_n \geq m-4$ *becuase otherwise, the integers in* $[1, m-4]$ *could not all be represented by the (universal) leaf. So the coefficient condition (5.3.3) in Lemma 5.3.4 on the total sum of all coefficients*

$$a_1 + \cdots + a_n \geq m-4$$

is essential for any leaf $\sum_{i=1}^{n} a_i P_m(x_i)$*. By the Lemma 5.3.4, we may easily induce that the rank of a leaf* $\sum_{i=1}^{n} a_i P_m(\mathbf{x})$ *with (5.3.2) does not exceed* $m-4$ *since* $a_1 + \cdots + a_n \geq n$ *and (5.3.3) holds for* $n = m-4$ *only if* $a_1 = \cdots = a_{m-5} = 1$*. Since the truant of the node*

$$P_m(x_1) + \cdots + P_m(x_{m-5})$$

is $m-4$, it would have its children

$$P_m(x_1)+\cdots+P_m(x_{m-5})+a_{m-4}P_m(x_{m-4}) \tag{5.3.6}$$

where $1 \le a_{m-4} \le m-4$ and that's all. Conesequently, we may yield that a leaf

$$a_1P_m(x_1)+\cdots+a_nP_m(x_n)$$

with (5.3.2) has $n \le m-4$ and there are excatly $m-4$ leaves with (5.3.2) of the rank $n=m-4$ of (5.3.6).

Until now, we showed that the rank of a leaf, i.e.,universal escalator $\sum\limits_{i=1}^{n} a_iP_m(x_i)$ otherwise

$$a_3=a_4=\cdots=a_{(C-4)l_m+5}=3 \tag{5.3.7}$$

does not exceed $m-4$. In the Lemma 5.3.6 and Lemma 5.3.7, we finally treat the leaves $\sum\limits_{i=1}^{n} a_iP_m(x_i)$ with (5.3.7). Actually among the above leaves, the universal escaltors of the maximal rank R_m would appear.

Lemma 5.3.6. *Let $F_m(\mathbf{x})=\sum\limits_{i=1}^{n} a_iP_m(x_i)$ be a node of the escalator tree with*

$$\begin{cases} a_3=a_4=\cdots=a_{(C-4)l_m+5}=3 \\ a_i \equiv 0 \pmod 3 \text{ for all } 1 \le i \le n. \end{cases} \tag{5.3.8}$$

If

$$a_1+a_2+\cdots+a_{n-1} < 3(m-3)-1 \le a_1+a_2+\cdots+a_n, \tag{5.3.9}$$

then the escalator would be universal, i.e., the node would become a leaf of the tree.

Proof. For

$$b_i := \begin{cases} a_i & \text{for } i=1,2 \\ a_{i+5} & \text{for } i \ge 3, \end{cases}$$

similarly with the proof of Lemma 5.3.4, we may get that the m-gonal form

$f_m(\mathbf{x}) := \sum_{i=1}^{n-5} b_i P_m(x_i)$ may represent every positive integer up to $3(m-2)$ except at most 5 positive integers in $[3(C-2)l_m+3, 3(m-3)-1]$ and these integers would have the same residue modulo 3 and the congruent integers would be consecutive. On the specific, the cases that there is an integer in $[3(C-2)l_m+3, 3(m-3)-1]$ not represented by $f_m(\mathbf{x})$ occur only when

$$\begin{cases} a_1+\cdots+a_n-15 = b_1+\cdots+b_{n-5} < 3(m-3)-1 \quad \text{or} \\ a_n > a_1+\cdots+a_{n-1}+1-15. \end{cases} \tag{5.3.10}$$

In the cases that $f_m(\mathbf{x})$ represents every positive integer up to $3(m-2)$, we may conclude that $F_m(\mathbf{x})$ is universal by using the fact that

$$a_3P_m(x_3)+\cdots+a_7P_m(x_7) = 3P_m(x_3)+\cdots+3P_m(x_7)$$

represents all the multiples of $3(m-2)$ from Lemma 5.2.1.

Now assume that there is an integer in $[3(C-2)l_m+3, 3(m-3)-1]$ not represented by $f_m(\mathbf{x})$ and let

$$E_1 < E_2(=E_1+3) < \cdots < E_s(=E_1+3(s-1))$$

where $1 \le s \le 5$ be all of the positive integers not represented by $f_m(\mathbf{x})$ in $[1, 3(m-3)-1]$. Through similar arguments with the proof of Lemma 5.3.4, we may obtain that $F_m(\mathbf{x})$ represents every positive integer except the below at most 9 integers

$$\begin{array}{ccccc} E_1 & E_2 & \cdots & E_{s-1} & E_s \\ & E_2+(m-2) & \cdots & E_{s-1}+(m-2) & \\ & & E_3+2(m-2) & & \end{array} \tag{5.3.11}$$

by using the fact that $a_3P_m(x_3)+\cdots+a_7P_m(x_7)$ represents all the multiples of $3(m-2)$ from Lemma 5.2.1.

On the other hand, we may directly check that the above integers in (5.3.11) are represented by $F_m(\mathbf{x})$ similarly with the argument of the proof of Lemma 5.3.4. $\square$

Lemma 5.3.7. *Let* $F_m(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ *be a node of the escalator tree with*

$$\begin{cases} a_3 = a_4 = \cdots = a_{(C-4)l_m+5} = 3 \\ a_i \equiv 0 \pmod 3 \text{ for all } 1 \le i \le n-1. \end{cases} \tag{5.3.12}$$

If $a_n \not\equiv 0 \pmod 3$*, then the escalator would be universal, i.e., the node would become a leaf of the tree.*

Proof. In virtue of the Lemma 5.3.6, we may have that $a_1 + \cdots + a_{n-1} < 3(m-3) - 1$. And then we may see that the truant of $\sum_{i=1}^{n-1} a_i P_m(x_i)$ would be one of

$$a_1 + \cdots + a_{n-1} + 1, a_1 + \cdots + a_{n-1} + 2, \text{ or } a_1 + \cdots + a_{n-1} + 3,$$

which implies that

$$a_n \le a_1 + \cdots + a_{n-1} + 3 < 3(m-3) + 2.$$

For

$$b_i := \begin{cases} a_i & \text{for } i = 1, 2 \\ a_{i+5} & \text{for } i \ge 3, \end{cases}$$

similarly with the proof of Lemma 5.3.4, we may obtain that $f_m(\mathbf{x}) := \sum_{i=1}^{n-5} b_i P_m(x_i)$ represents every positive integer up to $3(m-2)$ except at most 5 positive integers in $[3(C-2)l_m + 3, 3(m-3) - 1]$ and these integers would have the same residue modulo 3 and the congruent integers would be consecutive. On the specific, the cases that there is an integer in $[3(C-2)l_m + 3, 3(m-3) - 1]$ not represented by $f_m(\mathbf{x})$ occur only when

$$a_n > a_1 + \cdots + a_{n-1} + 1 - 15. \tag{5.3.13}$$

In the case that $f_m(\mathbf{x})$ represents every positive integer up to $3(m-2)$, we may conclude that the $F_m(\mathbf{x})$ is universal by using the fact that $a_3 P_m(x_1) + \cdots + a_7 P_m(x_7)$ represents all the multiples of $3(m-2)$ from Lemma 5.2.1.

Now assume that there is an integer in $[3(C-2)l_m + 3, 3(m-3) - 1]$ not represented by $f_m(\mathbf{x})$ and let

$$E_1 < E_2(= E_1 + 3) < \cdots < E_s(= E_1 + 3(s-1))$$

where $1 \le s \le 5$ be all of the positive integers not represented by $f_m(\mathbf{x})$ in $[3(C-2)l_m + 3, 3(m-3)-1]$. Through similar arguments with the proof of Lemma 5.3.4, we may obtain that $F_m(\mathbf{x})$ represents every positive integer except the below at most 9 integers

$$\begin{array}{ccccc} E_1 & E_2 & \cdots & E_{s-1} & E_s \\ & E_2 + (m-2) & \cdots & E_{s-1} + (m-2) & \\ & & E_3 + 2(m-2) & & \end{array} \tag{5.3.14}$$

in virtue of the fact that $a_3P_m(x_3) + \cdots + a_7P_m(x_7)$ represents all the multiples of $3(m-2)$ from Lemma 5.2.1.

On the other hand, we may directly check that the above integers in (5.3.14) are represented by $F_m(\mathbf{x})$ similarly with the argument of the proof of Lemma 5.3.4. □

Theorem 5.3.8. *For $m > 6C^2(C+1)$,*

$$R_m = \begin{cases} m-2 & \text{when } m \not\equiv 2 \pmod 3 \\ m-3 & \text{when } m \equiv 2 \pmod 3. \end{cases}$$

Proof. From Lemma 5.3.6 and Lemma 5.3.7, we may yield that the rank of a leaf $\sum\limits_{i=1}^{n} a_iP_m(x_i)$ with

$$a_3 = a_4 = \cdots = a_{(C-4)l_m+5} = 3 \tag{5.3.15}$$

would not exceed $m-2$ because $a_1 + a_2 + a_3 + \cdots + a_n \ge 2 + 3(n-2)$ under (5.3.15). Overall, with the Remark 5.3.5, we may get an upper bound $m-2$ for the rank of universal escalator.

On the other hand, we may have that the below escalators

$$\begin{cases} P_m(x_1)+P_m(x_2)+\sum_{i=3}^{m-3} 3P_m(x_i) & \text{when } m \equiv 0 \pmod 3 \\ P_m(x_1)+P_m(x_2)+\sum_{i=3}^{m-3} 3P_m(x_i) & \text{when } m \equiv 1 \pmod 3 \\ P_m(x_1)+2P_m(x_2)+\sum_{i=3}^{m-4} 3P_m(x_i) & \text{when } m \equiv 2 \pmod 3 \end{cases} \tag{5.3.16}$$

are not universal with the truant $3m-10, 3m-12$, and $3m-12$, respectively and we may see that from Lemma 5.3.6 and Lemma 5.3.7, they are all of the non-universal escalators of the rank greater than or equal to $m-3, m-3$, and $m-4$, respectively because

$$\begin{cases} P_m(x_1)+2P_m(x_2)+\sum_{i=3}^{\frac{2m-3}{3}} 3P_m(x_i) & \text{when } m \equiv 0 \pmod 3 \\ P_m(x_1)+2P_m(x_2)+\sum_{i=3}^{\frac{2m-5}{3}} 3P_m(x_i) & \text{when } m \equiv 1 \pmod 3 \\ P_m(x_1)+P_m(x_2)+\sum_{i=3}^{\frac{2m-4}{3}} 3P_m(x_i) & \text{when } m \equiv 2 \pmod 3 \end{cases} \tag{5.3.17}$$

are universal. This completes the proof of this theorem. $\square$

Remark 5.3.9. *In the Chapter 4 and Chapter 5, we proved that for* $m > 2\left(\left(2C+\frac{1}{4}\right)^{\frac{1}{4}}+\sqrt{2}\right)^2$,

$$r_m = \begin{cases} \lceil \log_2(m-3) \rceil + 1 & \textit{when } -3 \le 2^{\lceil \log_2(m-3) \rceil} - m \le 1 \\ \lceil \log_2(m-3) \rceil & \textit{when } \quad 2 \le 2^{\lceil \log_2(m-3) \rceil} - m \end{cases}$$

and for $m > 6C^2(C+1)$,

$$R_m = \begin{cases} m-2 & \textit{when } m \not\equiv 2 \pmod 3 \\ m-3 & \textit{when } m \equiv 2 \pmod 3. \end{cases}$$

Especially, we showed that

$$F_m^{(0)}(\mathbf{x}) := P_m(x_1) + 2P_m(x_2) + \cdots + 2^{r_m-1}P_m(x_{r_m})$$

is a leaf of the minimal rank r_m, i.e., universal m-gonal form of the minimal rank r_m.

Now for $m > 2\left(\left(2C+\frac{1}{4}\right)^{\frac{1}{4}}+\sqrt{2}\right)^2$, with $2 \le 2^{\lceil \log_2(m-3)\rceil} - m$, consider a sequence $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4-r_m}$ of leaves(universal escalators) which is inductively constructed from $F_m^{(0)}(\mathbf{x})$ by splitting the first component $a_iP_m(x_i)$ with non-one coefficient $a_i(>1)$ into two components $P_m(\cdot)$ and $(a_i-1)P_m(\cdot)$, namely, from the $(j+1)$-th leaf of $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4-r_m}$

$$F_m^{(j)}(\mathbf{x}) := P_m(x_1) + \cdots + P_m(x_{i-1}) + a_iP_m(x_i) + \cdots + a_nP_m(x_n)$$

with $a_i > 1$ the $(j+2)$-th leaf is constructed as

$$F_m^{(j+1)}(\mathbf{x}) := P_m(x_1)+\cdots+P_m(x_{i-1})+P_m(x_i)+(a_i-1)P_m(x_{i+1})+\cdots+a_nP_m(x_{n+1})$$

by rearranging the variables. By using the notation $\mathcal{C}_j := [a_1, \cdots, a_n]$ for $F_m^{(j)}(\mathbf{x}) = \sum\limits_{i=1}^{n} a_iP_m(x_i)$, we may describe the stream of coefficients of the sequence $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4-r_m}$ as follows

$$\begin{array}{rllllllll}
\mathcal{C}_0 = & [1, & 2, & 4, & 8, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\mathcal{C}_1 = & [1, & 1,1, & 4, & 8, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\mathcal{C}_2 = & [1, & 1,1, & 1,3, & 8, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\mathcal{C}_3 = & [1, & 1,1, & 1,1,2, & 8, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\mathcal{C}_4 = & [1, & 1,1, & 1,1,1,1, & 8, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\mathcal{C}_5 = & [1, & 1,1, & 1,1,1,1, & 1,7, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\mathcal{C}_6 = & [1, & 1,1, & 1,1,1,1, & 1,1,6, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\mathcal{C}_7 = & [1, & 1,1, & 1,1,1,1, & 1,1,1,5, & 16, & \cdots, & 2^{r_m-2}, & 2^{r_m-1}] \\
\vdots & & & & \vdots & & & & .
\end{array}$$

Then we may see rank $F_m^{(j)}(\mathbf{x}) = r_m + j$ ultimately there appear universal *escalators of rank from r_m and to $m-4$ in the sequence $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4-r_m}$.*

On the other hand for $m > 2\left(\left(2C+\frac{1}{4}\right)^{\frac{1}{4}}+\sqrt{2}\right)^2$ with $-3 \le 2^{\lceil \log_2(m-3)\rceil} - m \le 1$, even though every m-gonal form of the inductively constructed sequence $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4+r_m}$ as the same manner with the above is universal, but not all of them are proper universal, i.e., for some $F_m^{(j)}(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$, its proper subform

$$F_m^{(j)}(\mathbf{x}) - 2^{r_m-1}P_m(x_n)$$

would also be universal and the subform would be indeed a leaf(i.e., proper univeral m-gonal form). For the smallest J for which $F_m^{(J)}(\mathbf{x})$ is not a leaf, by replacing the j-th m-gonal form $F_m^{(j)}(\mathbf{x}) = \sum\limits_{i=1}^{n} a_i P_m(x_i)$ for all where $j \ge J$ of the above sequence $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4-r_m}$ as its subform

$$F_m^{(j)}(\mathbf{x}) - 2^{r_m-1}P_m(x_n)$$

which is a leaf, we may get a sequence of leaf $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4-r_m}$ with

$$\text{rank } F_m^{(j)}(\mathbf{x}) = \begin{cases} r_m + j & \text{when } j < J \\ r_m + j - 1 & \text{when } j \ge J. \end{cases}$$

So in this case, there appear leaves of rank from r_m and to $m-5$ in the sequence $\{F_m^{(j)}(\mathbf{x})\}_{j=0}^{m-4-r_m}$. And $P_m(x_1)+\cdots+P_m(x_{m-4})$ is a leaf of the rank $m-4$.

When $m > 6C^2(C+1)$ with $m \not\equiv 2 \pmod 3$, we may show that

$$P_m(x_1)+P_m(x_2)+3P_m(x_3)+\cdots+3P_m(x_{r_m-2})+6P_m(x_{r_m-1})$$

is a leaf of rank $r_m - 1 = m-3$ by using the Lemma 5.3.6.

From the above arguments, we may see that for $m > 6C^2(C+1)$, there is a universal escalator of rank n for each $r_m \le n \le R_m$.

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국문초록

이 논문에서는 보편 m-각수형식에 대해서 공부한다.

3장에서는 이 논문의 주요 결과로써 m-각수형식의 보편성을 판별하는 유한집합 $[1,\gamma_m]$에 대해서 공부하고, 점진적으로 증가하는 γ_m의 증가속도가 m에 대해 정확하게 선형임을 증명한다.

4장에서는 충분히 큰 모든 m에 대하여 보편 m-각수형식을 가질 최소한의 차원 r_m 을 찾고 r_m을 차원으로 갖는 보편 m-각수 형식 중 특별한 형식을 공부한다.

5장에서는 보편 m-각수 에스컬레이터의 최대 차원을 공부한다.

주요어휘: 보편 m각수의 합
학번: 2014-21209

감사의 글

가장 먼저 제가 이차형식을 공부할 수 있게 지도해 주신 제가 너무나 존경하는 저의 지도교수님이신 김명환 선생님께 감사의 말씀을 드립니다. 많이 부족하고 모자란 저를 늘 진심 어린 관심과 크신 은혜로 보듬어 주셨습니다. 학문적으로 저의 어떤 질문에도 늘 진지하게 대답해 주시고 같이 고민해 주셨으며 선생님께서 주신 아낌없는 지원과 격려 덕분에 수학자로서 많은 경험을 하고 배울 수 있었습니다.

바쁘신 중에 멀리서도 일주일에 한번씩 시간을 내어 지도해 주신 강릉원주대학교 김병문 선생님께도 감사의 말씀을 드립니다. 선생님께서는 저에게 너무나 많은 수학적 영감을 주셨고 그 덕분에 제가 정말 학문(이차형식)의 즐거움을 깨닫게 되었습니다.

너무나 값진 두 분 선생님의 관심과 지도 덕분에 제가 이 논문을 쓸 수 있었습니다. 제가 너무 존경하고 좋아하는 훌륭하신 두 분 선생님의 지도를 받으며 공부할 수 있었던 시간들이 너무 행복했습니다. 이런 소중한 시간과 귀한 지도를 저에게 선물해 주신 두 분 선생님께 아무리 해도 부족한 말이지만 다시 한번 감사의 말씀을 올립니다.

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학회에서 짧게 뵈었지만 저의 이야기에 관심을 가져주시고 진지한 조언과 저의 연구에 아낌없는 격려를 해주신 A. G. Earnest 교수님께도 감사드립니다.

또 바쁘신 중에 저의 졸업 논문을 심사해 주시고 조언을 주신 오병권 교수님, 천정희 교수님, 권재훈 교수님께도 감사의 말씀을 드립니다.

항상 모자란 저를 챙겨주었던 김지승, 진지하게 고민을 들어주고 늘 현명한 제안을 해준 민현이 오빠, 늘 친절히 영어 작문을 도와준 혜림이, 문제를 해결하는데 큰 도움(프로그램)을 주신 김원세 조교님, 그 외에 같이 공부하며 힘이 되어준 모든 동기, 선배님, 후배님에게 감사드립니다.

마지막으로 항상 저를 믿고 저의 꿈을 응원해주시고 아낌없는 지원을 주시는 아직까지도 걱정만 끼쳐드리는 저의 부모님께 감사의 말씀을 올립니다.